Mathematics 30-1 & Pre-Calculus 12

Book One

Polynomial, Radical, and Rational Functions
Transformations and Operations
Exponential and Logarithmic Functions

A workbook and animated series by Barry Mabillard

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Mathematics 30-1 & Pre-Calculus 12

Formula Sheet

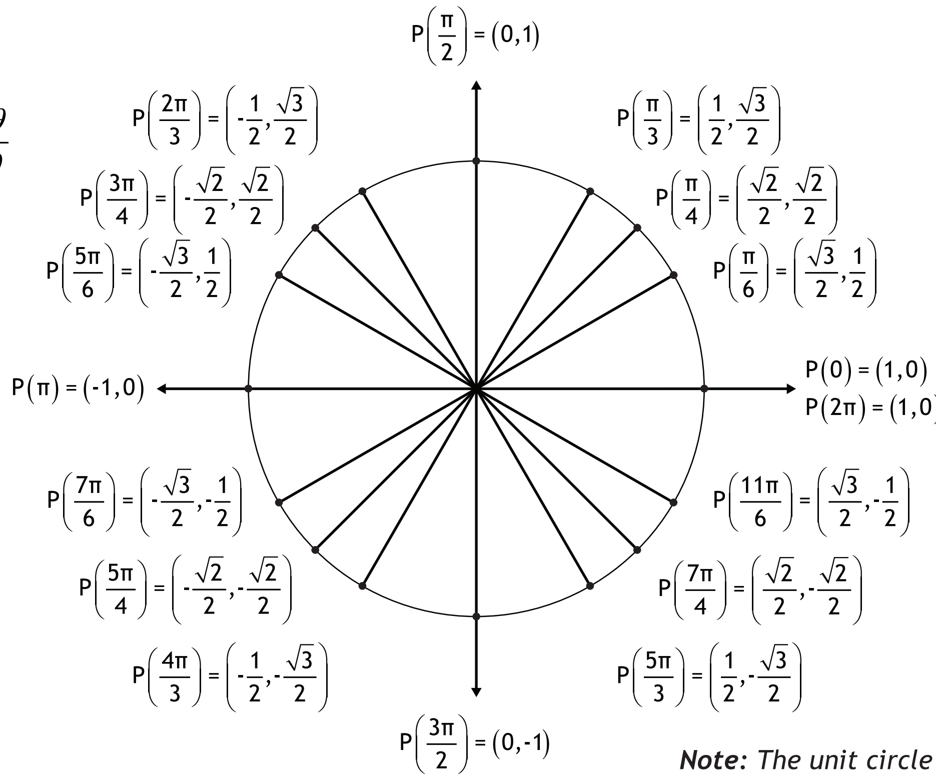
Trigonometry I

$$\theta = \frac{a}{r}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

The Unit Circle



***Note:** The unit circle is **NOT** included on the official formula sheet.*

Trigonometry II

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin(2A) = 2\sin A\cos A$$

$$\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$$

Transformations & Operations

$$y = af\left[b(x-h)\right] + k$$

Polynomial, Radical & Rational Functions

$$x : \left[x_{\min}, x_{\max}, x_{\mathrm{scl}}\right]$$

$$y : \left[y_{\min}, y_{\max}, y_{\mathrm{scl}}\right]$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exponential and Logarithmic Functions

$$\log_b(M \times N) = \log_b M + \log_b N$$

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

$$\log_b\left(M^n\right) = n\log_b M$$

$$\log_b c = \frac{\log_a c}{\log_a b}$$

$$y = ab^{\frac{t}{P}}$$

Permutations & Combinations

$$n! = n(n-1)(n-2)\ldots 3 \times 2 \times 1$$

$${}_nP_r = \frac{n!}{(n-r)!}$$

$${}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

$$t_{k+1} = {}_nC_k x^{n-k} y^k$$

Curriculum Alignment
Math 30-1: Alberta | Northwest Territories | Nunavut
Pre-Calculus 12: British Columbia | Yukon
Pre-Calculus 30: Saskatchewan
Pre-Calculus 40S: Manitoba

Mathematics 30-1 & Pre-Calculus 12

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Example 1

Introduction to Polynomial Functions.

Defining Polynomials

a) Given the general form of a polynomial function:

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x^1 + a_0$$

the leading coefficient is ______.

the degree of the polynomial is ______.

the constant term of the polynomial is ______.

For each polynomial function given below, state the leading coefficient, degree, and constant term.

i) $f(x) = 3x - 2$

leading coefficient: ______ degree: ______ constant term: ______

ii) $y = x^3 + 2x^2 - x - 1$

leading coefficient: ______ degree: ______ constant term: ______

iii) $P(x) = 5$

leading coefficient: ______ degree: ______ constant term: ______

b) Determine which expressions are polynomials. Explain your reasoning.

i) $x^5 + 3$

polynomial: yes no

ii) $5^x + 3$

polynomial: yes no

iii) 3

polynomial: yes no

iv) $4x^2 - 5x^{\frac{1}{2}} - 1$

polynomial: yes no

v) $x^2 + \frac{1}{3}x - 4$

polynomial: yes no

vi) $|x|$

polynomial: yes no

vii) $5\sqrt{x} - 1$

polynomial: yes no

viii) $\sqrt{7}x + 2$

polynomial: yes no

ix) $\frac{1}{x + 3}$

polynomial: yes no

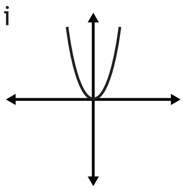
Example 2

End Behaviour of Polynomial Functions.

Even-Degree Polynomials

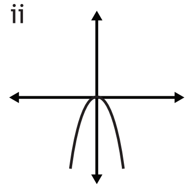
a) The equations and graphs of several even-degree polynomials are shown below. Study these graphs and generalize the end behaviour of even-degree polynomials.

i



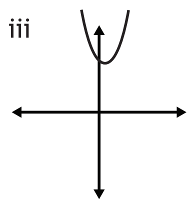
$f(x) = x^2$
quadratic

ii



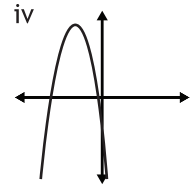
$f(x) = -x^2$
quadratic

iii



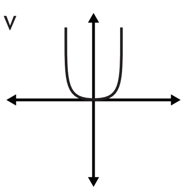
$f(x) = x^2 - x + 6$
quadratic

iv



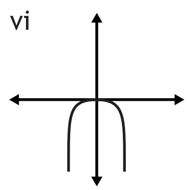
$f(x) = -x^2 - 8x - 7$
quadratic

v



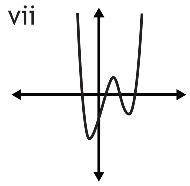
$f(x) = x^4$
quartic

vi



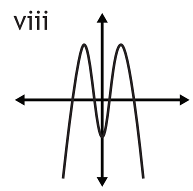
$f(x) = -x^4$
quartic

vii



$f(x) = x^4 - 4x^3 + x^2 + 7x - 3$
quartic

viii



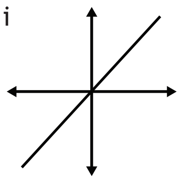
$f(x) = -x^4 + 7x^2 - 5$
quartic

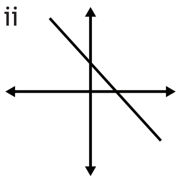
End behaviour of even-degree polynomials:

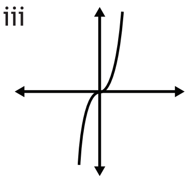
Sign of Leading Coefficient	*End Behaviour*
Positive	
Negative	

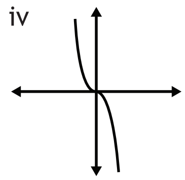
b) The equations and graphs of several odd-degree polynomials are shown below. Study these graphs and generalize the end behaviour of odd-degree polynomials.

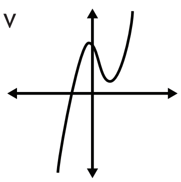
Odd-Degree Polynomials

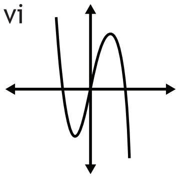
i

$f(x) = x$
linear

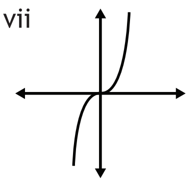
ii

$f(x) = -x + 4$
linear

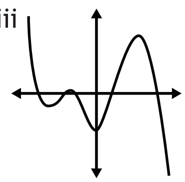
iii

$f(x) = x^3$
cubic

iv

$f(x) = -x^3$
cubic

v

$f(x) = x^3 - 2x^2 - 2x + 6$
cubic

vi

$f(x) = -x^3 + 7x$
cubic

vii

$f(x) = x^5$
quintic

viii

$f(x) = -x^5 - 4x^4 + 40x^3 + 160x^2 - 144x - 576$
quintic

End behaviour of odd-degree polynomials:

Sign of Leading Coefficient	*End Behaviour*
Positive	
Negative	

Example 3

Zeros, Roots, and x-intercepts of a Polynomial Function.

Zeros, roots, and x-intercepts

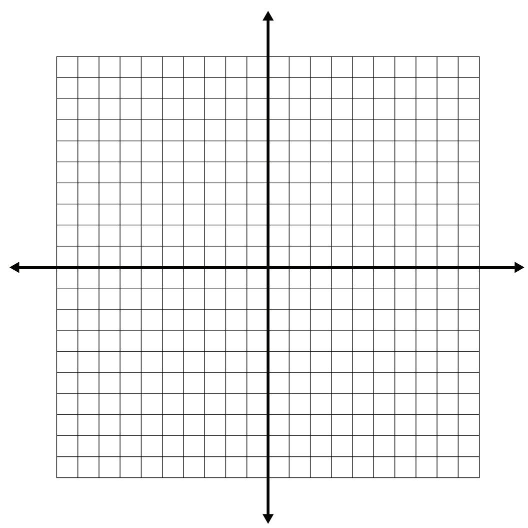
a) Define "*zero of a polynomial function*". Determine if each value is a zero of $P(x) = x^2 - 4x - 5$.

i) -1

ii) 3

b) Find the zeros of $P(x) = x^2 - 4x - 5$ by solving for the roots of the related equation, $P(x) = 0$.

c) Use a graphing calculator to graph $P(x) = x^2 - 4x - 5$.
How are the zeros of the polynomial related to the x-intercepts of the graph?



d) How do you know when to describe solutions as zeros, roots, or x-intercepts?

Example 4

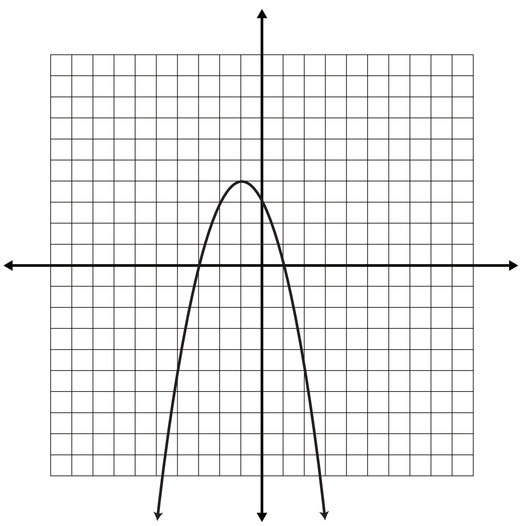
Multiplicity of Zeros in a Polynomial Function.

Multiplicity

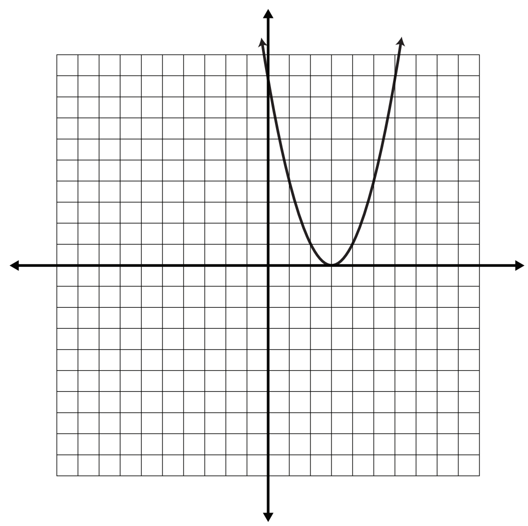
a) Define "*multiplicity of a zero*".

For the graphs in parts (b - e), determine the zeros and state each zero's multiplicity.

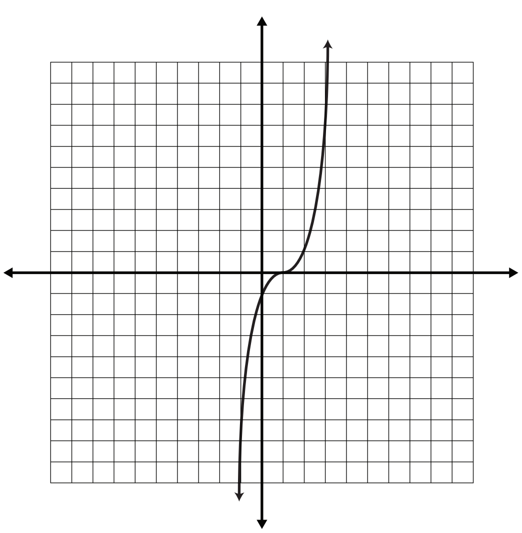
b) $P(x) = -(x + 3)(x - 1)$



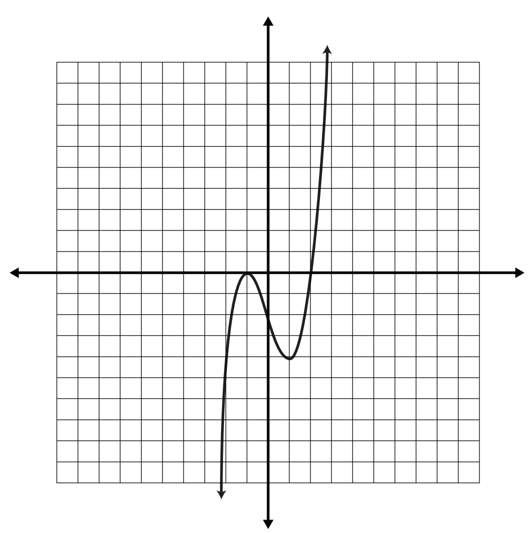
c) $P(x) = (x - 3)^2$



d) $P(x) = (x - 1)^3$



e) $P(x) = (x + 1)^2(x - 2)$

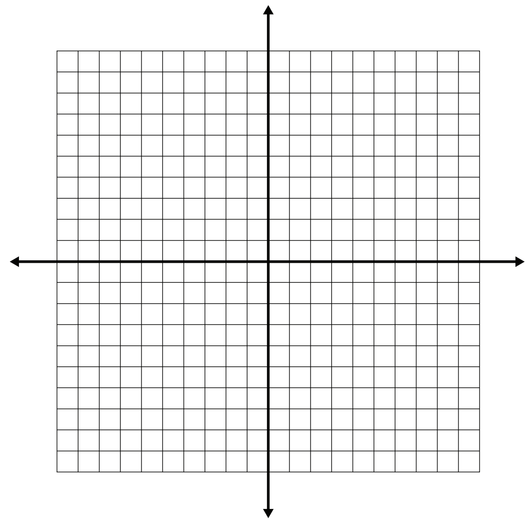


Example 5 Find the requested data for each polynomial function, then use this information to sketch the graph.

Graphing Polynomials

a) $P(x) = \frac{1}{2}(x - 5)(x + 3)$ *Quadratic polynomial with a positive leading coefficient.*

i) Find the zeros and their multiplicities.



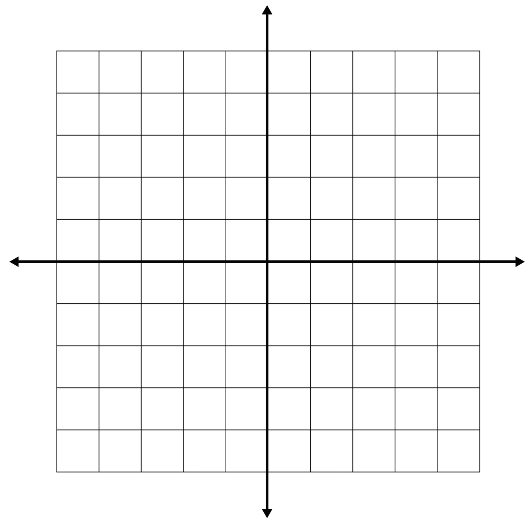
ii) Find the y-intercept.

iii) Describe the end behaviour.

iv) What other points are required to draw the graph accurately?

b) $P(x) = -x^2(x + 1)$ *Cubic polynomial with a negative leading coefficient.*

i) Find the zeros and their multiplicities.



ii) Find the y-intercept.

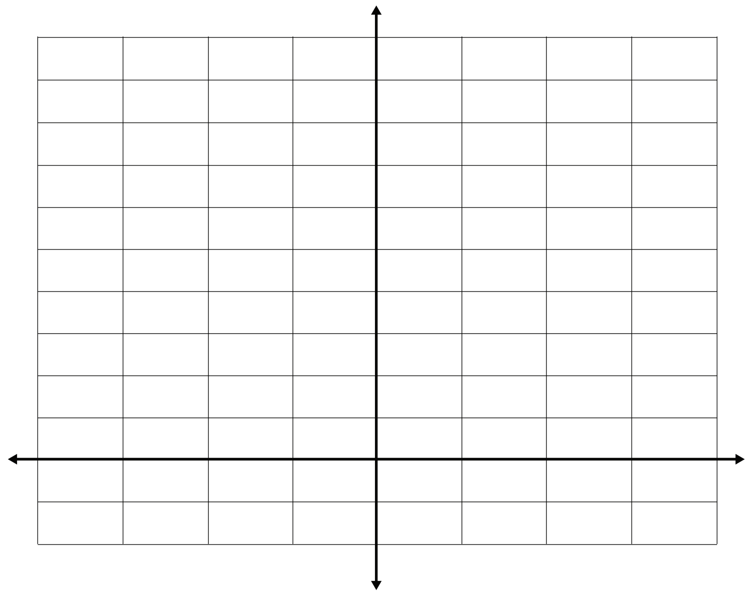
iii) Describe the end behaviour.

iv) What other points are required to draw the graph accurately?

Example 6 Find the requested data for each polynomial function, then use this information to sketch the graph.

Graphing Polynomials

a) $P(x) = (x - 1)^2(x + 2)^2$ *Quartic polynomial with a positive leading coefficient.*



i) Find the zeros and their multiplicities.

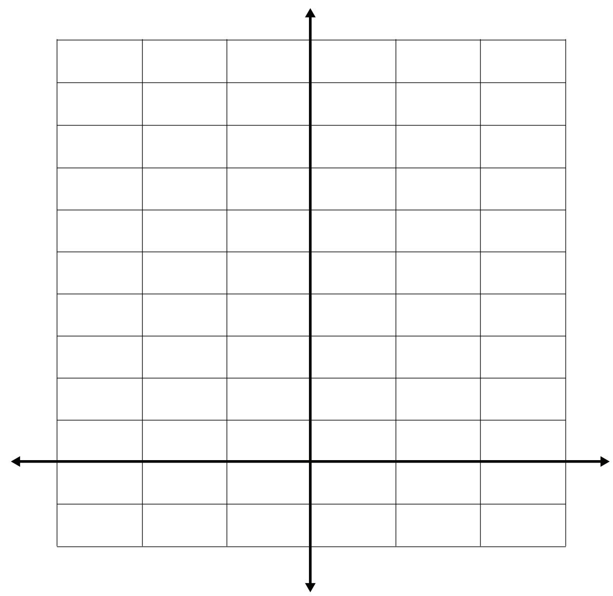
ii) Find the y-intercept.

iii) Describe the end behaviour.

iv) What other points are required to draw the graph accurately?

b) $P(x) = x(x + 1)^3(x - 2)^2$ *Sixth-degree polynomial with a positive leading coefficient.*

i) Find the zeros and their multiplicities.



ii) Find the y-intercept.

iii) Describe the end behaviour.

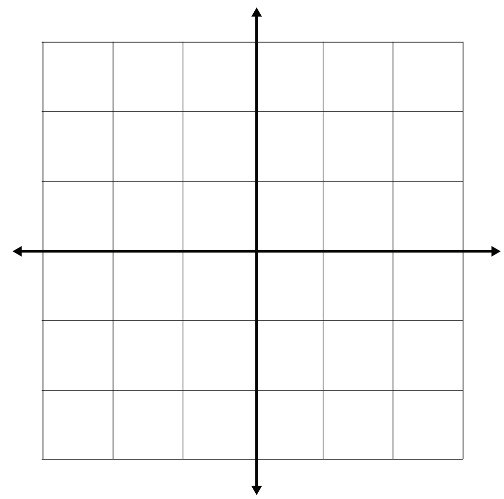
iv) What other points are required to draw the graph accurately?

Example 7

Find the requested data for each polynomial function, then use this information to sketch the graph.

Graphing Polynomials

a) $P(x) = -(2x - 1)(2x + 1)$ *Quadratic polynomial with a negative leading coefficient.*



i) Find the zeros and their multiplicities.

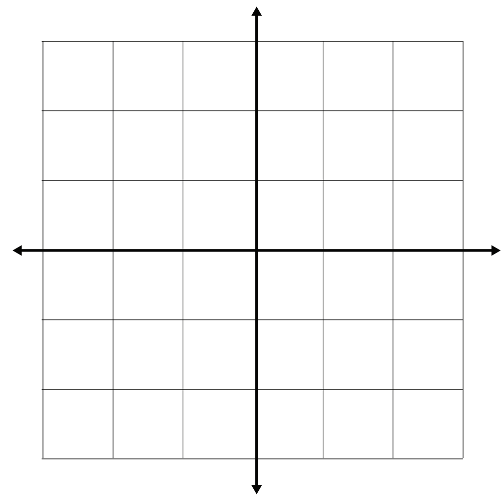
ii) Find the y-intercept.

iii) Describe the end behaviour.

iv) What other points are required to draw the graph accurately?

b) $P(x) = x(4x - 3)(3x + 2)$ *Cubic polynomial with a positive leading coefficient.*

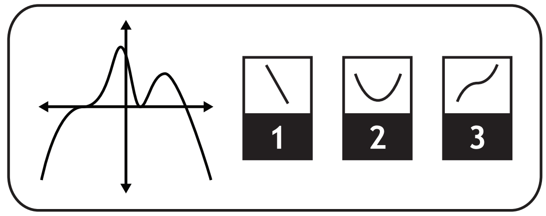
i) Find the zeros and their multiplicities.



ii) Find the y-intercept.

iii) Describe the end behaviour.

iv) What other points are required to draw the graph accurately?

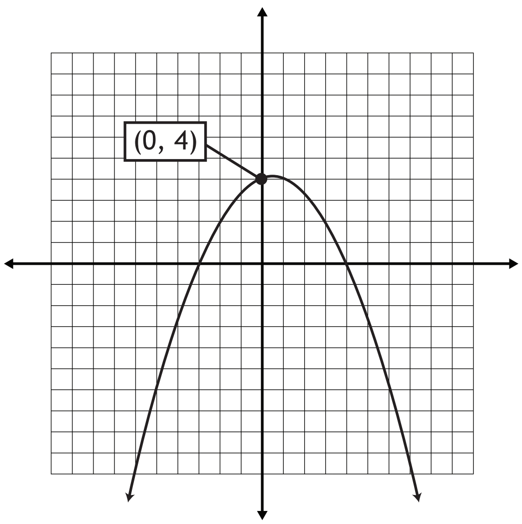


Example 8

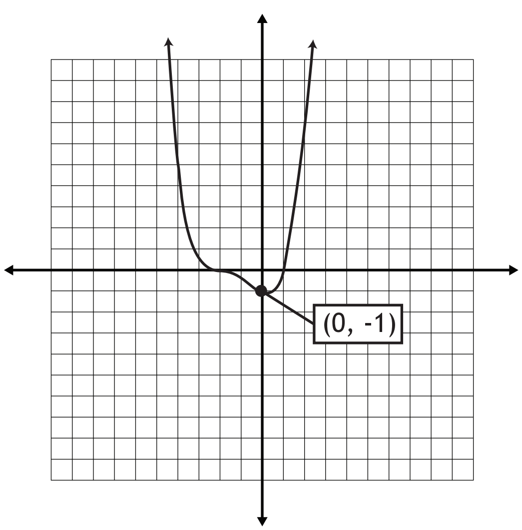
Determine the polynomial function corresponding to each graph. You may leave your answer in factored form.

Finding a Polynomial From its Graph

a)



b)

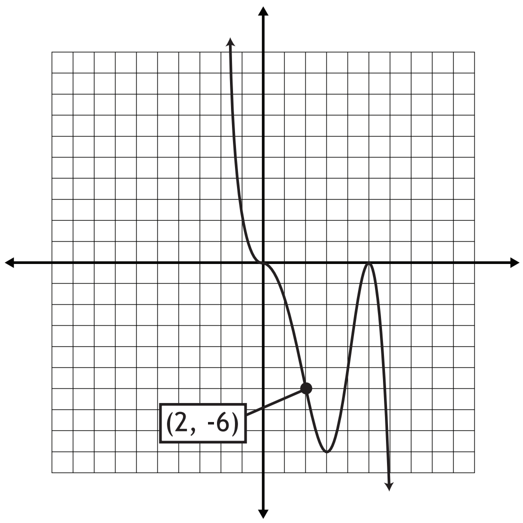


Example 9

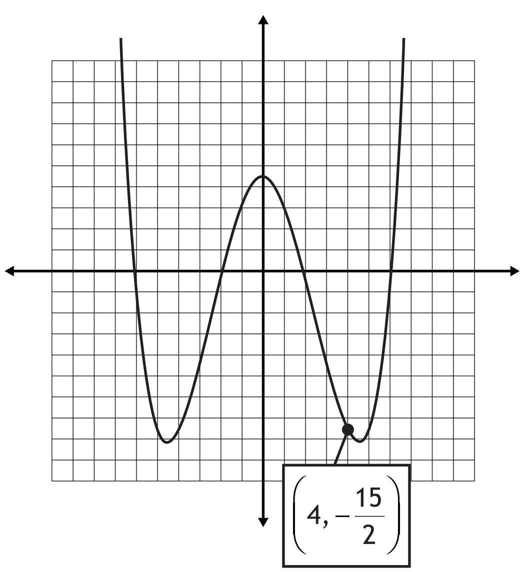
Determine the polynomial function corresponding to each graph. You may leave your answer in factored form.

Finding a Polynomial From its Graph

a)



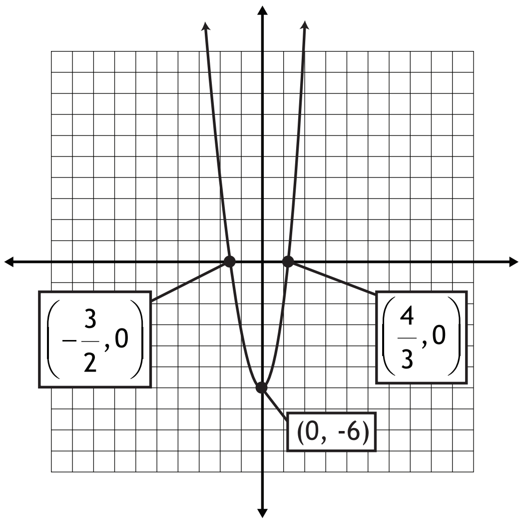
b)



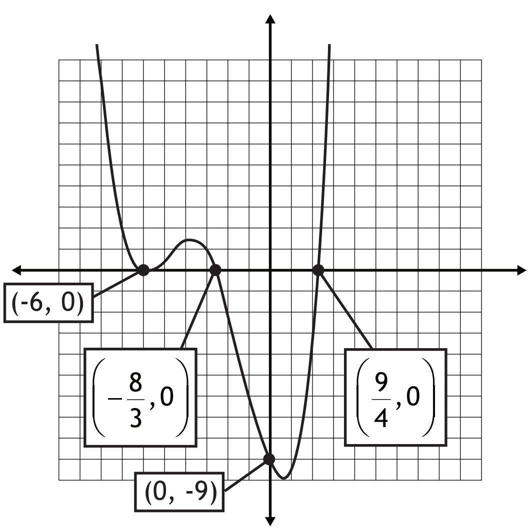
Example 10 Determine the polynomial function corresponding to each graph. You may leave your answer in factored form.

Finding a Polynomial From its Graph

a)



b)



Example 11

Use a graphing calculator to graph each polynomial function. Find window settings that clearly show the important features of each graph *(x-intercepts, y-intercept, and end behaviour)*.

Graphing Polynomials with Technology

a) $P(x) = x^2 - 2x - 168$

Draw the graph.

b) $P(x) = x^3 + 7x^2 - 44x$

Draw the graph.

c) $P(x) = x^3 - 16x^2 - 144x + 1152$

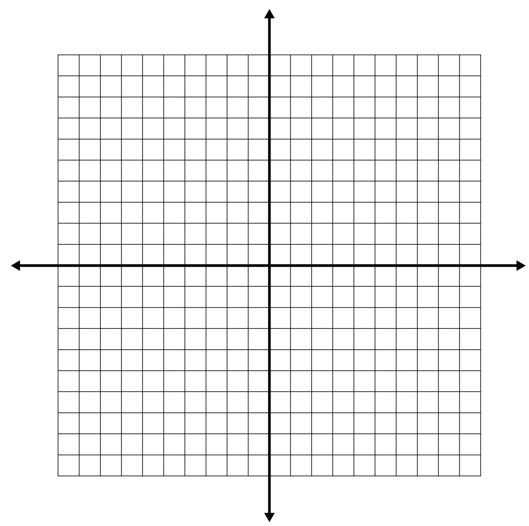
Draw the graph.

Example 12 Given the characteristics of a polynomial function, draw the graph and derive the actual function.

Graph and Write the Polynomial

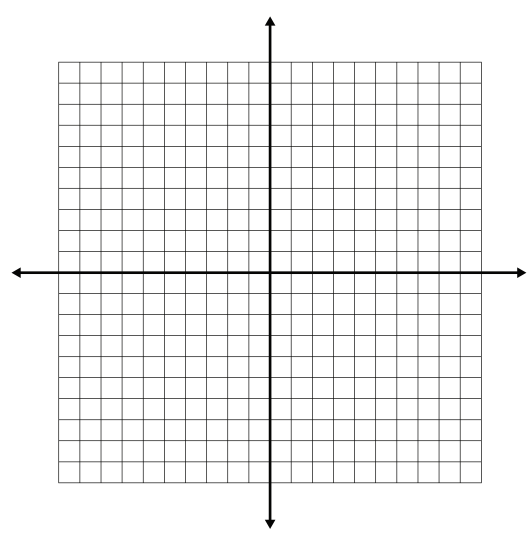
a) Characteristics of P(x):

x-intercepts: (-1, 0) and (3, 0)
sign of leading coefficient: (+)
polynomial degree: 4
relative maximum at (1, 8)



b) Characteristics of P(x):

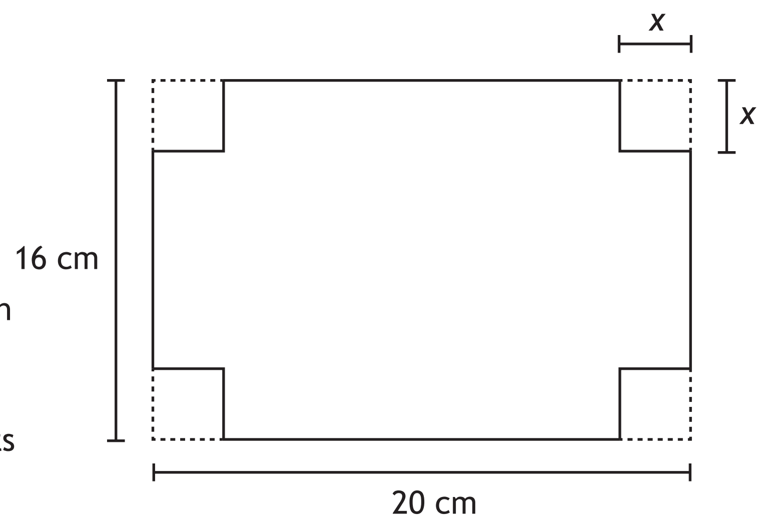
x-intercepts: (-3, 0), (1, 0), and (4, 0)
sign of leading coefficient: (-)
polynomial degree: 3
y-intercept at: $\left(0, -\frac{3}{2}\right)$

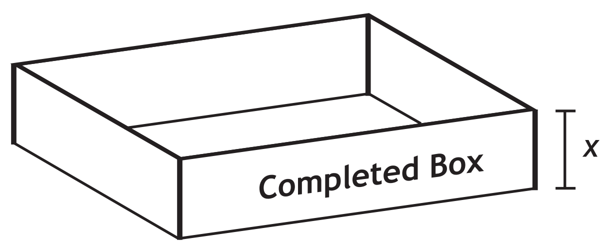


Example 13

A box with no lid can be made by cutting out squares from each corner of a rectangular piece of cardboard and folding up the sides.

A particular piece of cardboard has a length of 20 cm and a width of 16 cm. The side length of a corner square is x.





a) Derive a polynomial function that represents the volume of the box.

b) What is an appropriate domain for the volume function?

c) Use a graphing calculator to draw the graph of the function. Indicate your window settings.

Draw the graph.

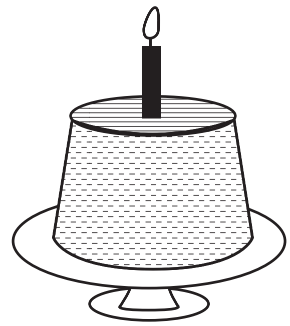
d) What should be the side length of a corner square if the volume of the box is maximized?

e) For what values of x is the volume of the box greater than 200 cm^3?

Draw the graph.

Example 14

Three students share a birthday on the same day. Quinn and Ralph are the same age, but Audrey is two years older. The product of their ages is 11548 greater than the sum of their ages.



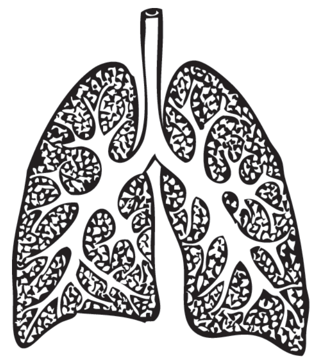
a) Find polynomial functions that represent the age product and age sum.

b) Write a polynomial equation that can be used to find the age of each person.

c) Use a graphing calculator to solve the polynomial equation from part (b). Indicate your window settings. How old is each person?

Draw the graph.

Example 15



The volume of air flowing into the lungs during a breath can be represented by the polynomial function $V(t) = -0.041t^3 + 0.181t^2 + 0.202t$, where V is the volume in litres and t is the time in seconds.

a) Use a graphing calculator to graph V(t). State your window settings.

Draw the graph.

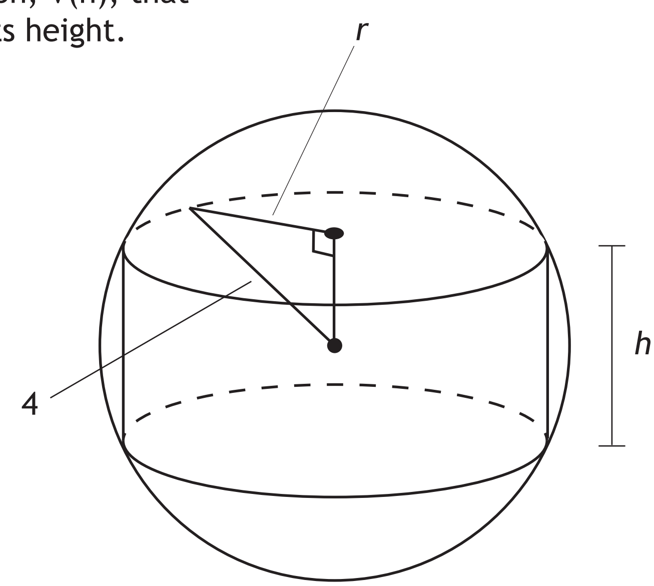
b) What is the maximum volume of air inhaled into the lung? At what time during the breath does this occur?

c) How many seconds does it take for one complete breath?

d) What percentage of the breath is spent inhaling?

Example 16

A cylinder with a radius of *r* and a height of *h* is inscribed within a sphere that has a radius of 4 units. Derive a polynomial function, V(h), that expresses the volume of the cylinder as a function of its height.



$$V_{cylinder} = \pi r^2 h$$

Lesson Notes

Example 1 Divide $(x^3 + 2x^2 - 5x - 6)$ by $(x + 2)$ using long division and answer the related questions.

Long & Synthetic Polynomial Division

a) $x + 2 \,\overline{)\, x^3 + 2x^2 - 5x - 6}$

b) Label the division components *(dividend, divisor, quotient, remainder)* in your work for part (a).

c) Express the division using the division theorem, $P(x) = Q(x) \bullet D(x) + R$. Verify the division theorem by checking that the left side and right side are equivalent.

1	3	-4	-5	2
-	↓	3	-7	2
	3	-7	2	0

d) Another way to represent the division theorem is $\frac{P(x)}{D(x)} = Q(x) + \frac{R}{D(x)}$.

Express the division using this format.

e) Synthetic division is a quicker way of dividing than long division. Divide $(x^3 + 2x^2 - 5x - 6)$ by $(x + 2)$ using synthetic division and express the result in the form $\frac{P(x)}{D(x)} = Q(x) + \frac{R}{D(x)}$.

Example 2

Divide using long division.

Express answers in the form $\frac{P(x)}{D(x)} = Q(x) + \frac{R}{D(x)}$.

Polynomial Division
(Long Division)

a) $(3x^3 - 4x^2 + 2x - 1) \div (x + 1)$

b) $\frac{x^3 - 3x - 2}{x - 2}$

c) $(x^3 - 1) \div (x + 2)$

$$\begin{array}{r|rrrr} 1 & 3 & -4 & -5 & 2 \\ - & \downarrow & 3 & -7 & 2 \\ \hline & 3 & -7 & 2 & 0 \end{array}$$

Polynomial Division
(Synthetic Division)

Example 3 Divide using synthetic division.

Express answers in the form $\frac{P(x)}{D(x)} = Q(x) + \frac{R}{D(x)}$.

a) $(3x^3 - x - 3) \div (x - 1)$

b) $\frac{3x^4 + 5x^3 + 3x - 2}{x + 2}$

c) $(2x^4 - 7x^2 + 4) \div (x - 1)$

Example 4

Polynomial division only requires long or synthetic division when factoring is not an option. Try to divide each of the following polynomials by factoring first, using long or synthetic division as a backup.

Polynomial Division
(Factoring)

a) $\dfrac{x^2 - 5x + 6}{x - 3}$

b) $(6x - 4) \div (3x - 2)$

c) $(x^4 - 16) \div (x^2 + 4)$

d) $\dfrac{x^3 + 2x^2 - 3x}{x - 3}$

Lesson Notes

1	3	-4	-5	2
-	↓	3	-7	2
	3	-7	2	0

Example 5

When $3x^3 - 4x^2 + ax + 2$ is divided by $x + 1$, the quotient is $3x^2 - 7x + 2$ and the remainder is zero. Solve for *a* using two different methods.

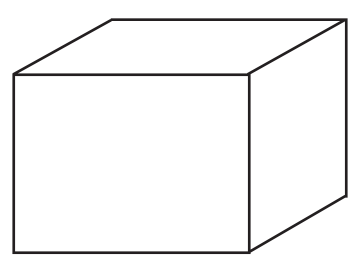
a) Solve for *a* using synthetic division.

b) Solve for *a* using $P(x) = Q(x) \bullet D(x) + R$.

Example 6

A rectangular prism has a volume of $x^3 + 6x^2 - 7x - 60$. If the height of the prism is $x + 4$, determine the dimensions of the base.

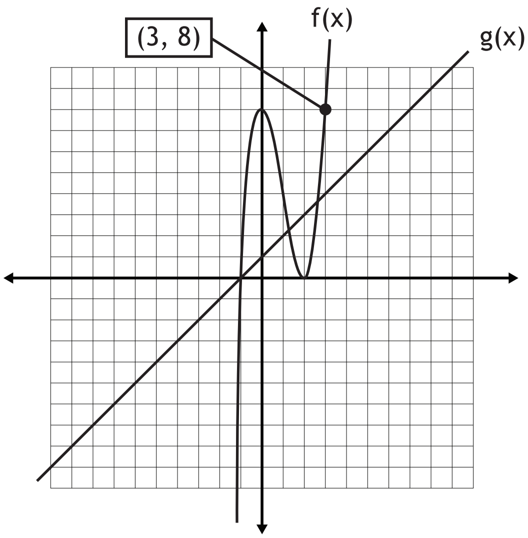
$V = x^3 + 6x^2 - 7x - 60$



$x + 4$

Example 7

The graphs of f(x) and g(x) are shown below.



a) Determine the polynomial corresponding to f(x).

b) Determine the equation of the line corresponding to g(x).

Recall that the equation of a line can be found using y = mx + b, where m is the slope of the line and the y-intercept is (0, b).

c) Determine Q(x) = f(x) ÷ g(x) and draw the graph of Q(x).

$$\begin{array}{c|cccc} 1 & 3 & -4 & -5 & 2 \\ - & \downarrow & 3 & -7 & 2 \\ \hline & 3 & -7 & 2 & 0 \end{array}$$

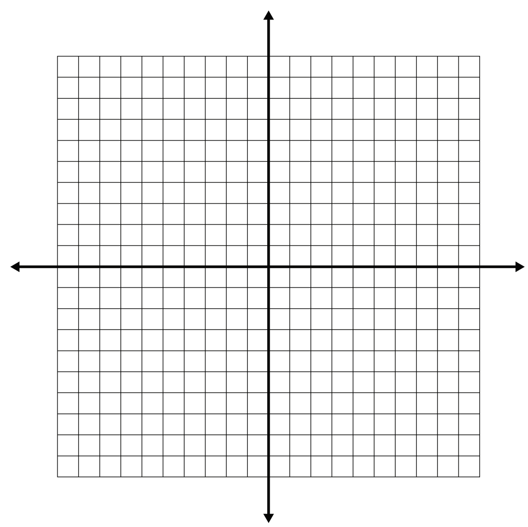
Example 8 If $f(x) \div g(x) = 4x^2 + 4x - 3 - \frac{6}{x - 1}$, determine f(x) and g(x).

Example 9 The Remainder Theorem

The Remainder Theorem

a) Divide $2x^3 - x^2 - 3x - 2$ by $x - 1$ using synthetic division and state the remainder.

b) Draw the graph of $P(x) = 2x^3 - x^2 - 3x - 2$ using technology.
What is the value of $P(1)$?



c) How does the remainder in part (a) compare with the value of $P(1)$ in part (b)?

d) Using the graph from part (b), find the remainder when $P(x)$ is divided by:

i) $x - 2$ ii) x iii) $x + 1$

e) Define the remainder theorem.

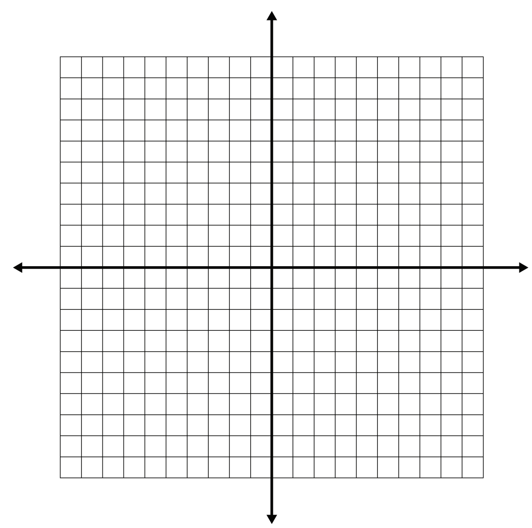
1	3	-4	-5	2
-	↓	3	-7	2
	3	-7	2	0

Example 10 The Factor Theorem

The Factor Theorem

a) Divide $x^3 - 3x^2 + 4x - 2$ by $x - 1$ using synthetic division and state the remainder.

b) Draw the graph of $P(x) = x^3 - 3x^2 + 4x - 2$ using technology.
What is the remainder when $P(x)$ is divided by $x - 1$?



c) How does the remainder in part (a) compare with the value of $P(1)$ in part (b)?

d) Define the factor theorem.

e) Draw a diagram that illustrates the relationship between the remainder theorem and the factor theorem.

Example 11

For each division, use the remainder theorem to find the remainder. Use the factor theorem to determine if the divisor is a factor of the polynomial.

Is the Divisor a Factor?

a) $(x^3 - 1) \div (x + 1)$

b) $\dfrac{x^4 - 2x^2 + 3x - 4}{x + 2}$

c) $(3x^3 + 8x^2 - 1) \div (3x - 1)$

d) $\dfrac{2x^4 + 3x^3 - 4x - 9}{2x + 3}$

1	3	-4	-5	2
-	↓	3	-7	2
	3	-7	2	0

Example 12

Use the remainder theorem to find the value of *k* in each polynomial.

One-Unknown Problems

a) $(kx^3 - x - 3) \div (x - 1)$ *Remainder = -1*

b) $\dfrac{3x^3 - 6x^2 + 2x + k}{x - 2}$ *Remainder = -3*

c) $(2x^3 + 3x^2 + kx - 3) \div (2x + 5)$ *Remainder = 2*

d) $\dfrac{2x^3 + kx^2 - x + 6}{2x - 3}$ *(2x - 3 is a factor)*

Example 13

Two-Unknown Problems

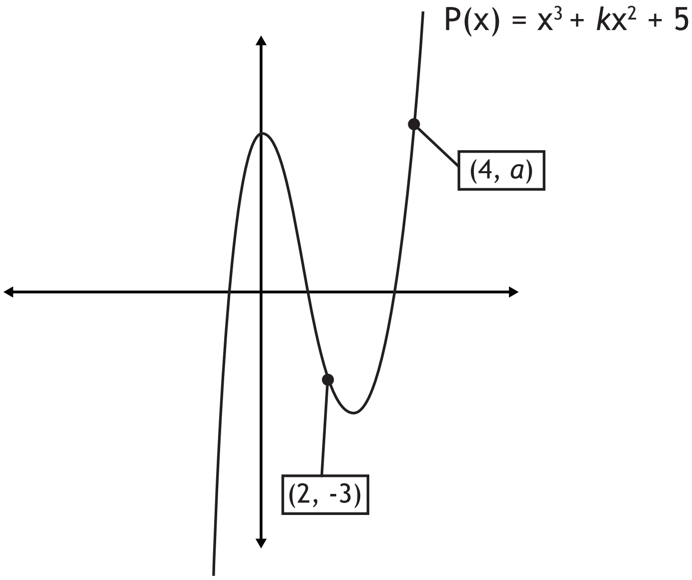
When $3x^3 + mx^2 + nx + 2$ is divided by $x + 2$, the remainder is 8. When the same polynomial is divided by $x - 1$, the remainder is 2. Determine the values of m and n.

Example 14

When $2x^3 + mx^2 + nx - 6$ is divided by $x - 2$, the remainder is 20. The same polynomial has a factor of $x + 2$. Determine the values of m and n.

Example 15

Given the graph of $P(x) = x^3 + kx^2 + 5$ and the point (2, -3), determine the value of *a* on the graph.



$x^3 - 5x^2 + 2x + 8$

$(x + 1)(x - 2)(x - 4)$

Polynomial, Radical, and Rational Functions
LESSON THREE - *Polynomial Factoring*
Lesson Notes

Example 1

The Integral Zero Theorem

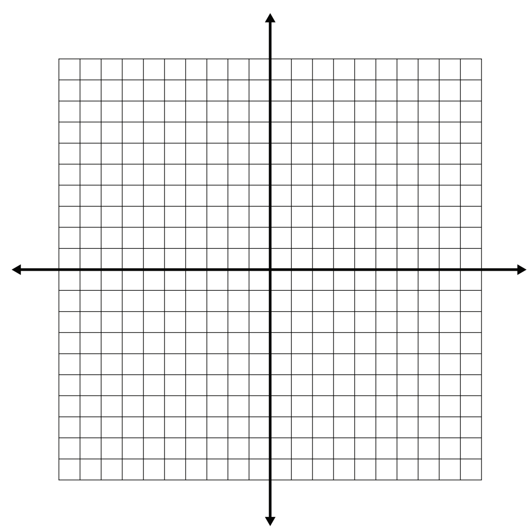
Integral Zero Theorem

a) Define the *integral zero theorem*. How is this theorem useful in factoring a polynomial?

b) Using the integral zero theorem, find potential zeros of the polynomial $P(x) = x^3 + x^2 - 5x + 3$.

c) Which potential zeros from part (b) are actually zeros of the polynomial?

d) Use technology to draw the graph of $P(x) = x^3 + x^2 - 5x + 3$. How do the x-intercepts of the graph compare to the zeros of the polynomial function?



e) Use the graph from part (d) to factor $P(x) = x^3 + x^2 - 5x + 3$.

Example 2

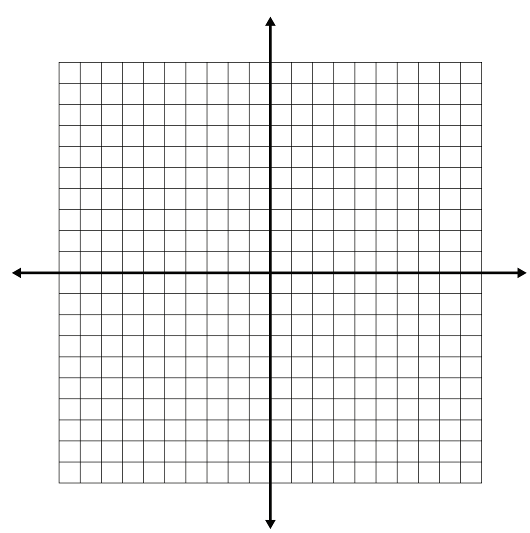
Factor and graph $P(x) = x^3 + 3x^2 - x - 3$.

Polynomial Factoring

a) Factor algebraically using the integral zero theorem.

b) Use technology to graph the polynomial. Can the polynomial be factored using just the graph?

c) Can P(x) be factored any other way?



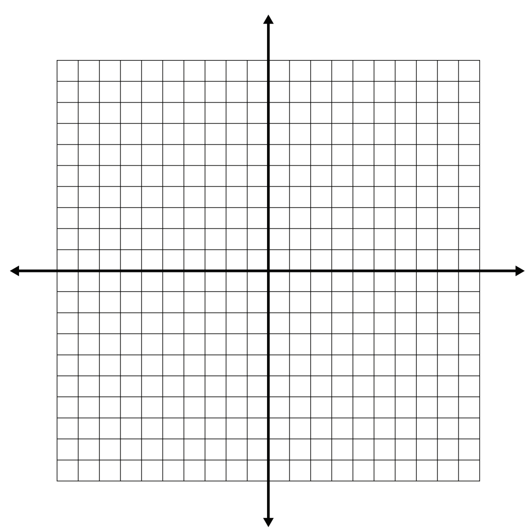
Example 3 Factor and graph $P(x) = 2x^3 - 6x^2 + x - 3$

Polynomial Factoring

a) Factor algebraically using the integral zero theorem.

b) Use technology to graph the polynomial. Can the polynomial be factored using just the graph?

c) Can P(x) be factored any other way?



Example 4

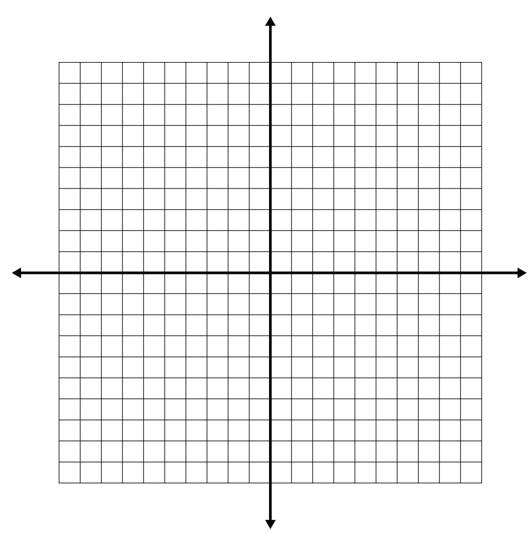
Factor and graph $P(x) = x^3 - 3x + 2$

Polynomial Factoring

a) Factor algebraically using the integral zero theorem.

b) Use technology to graph the polynomial. Can the polynomial be factored using just the graph?

c) Can $P(x)$ be factored any other way?



Example 5

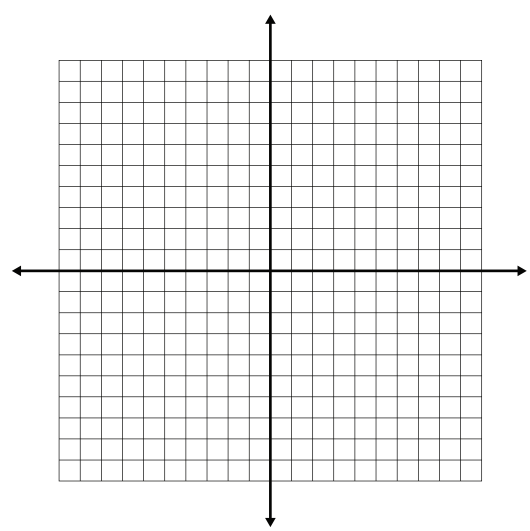
Factor and graph $P(x) = x^3 - 8$

Polynomial Factoring

a) Factor algebraically using the integral zero theorem.

b) Use technology to graph the polynomial. Can the polynomial be factored using just the graph?

c) Can $P(x)$ be factored any other way?



Example 6

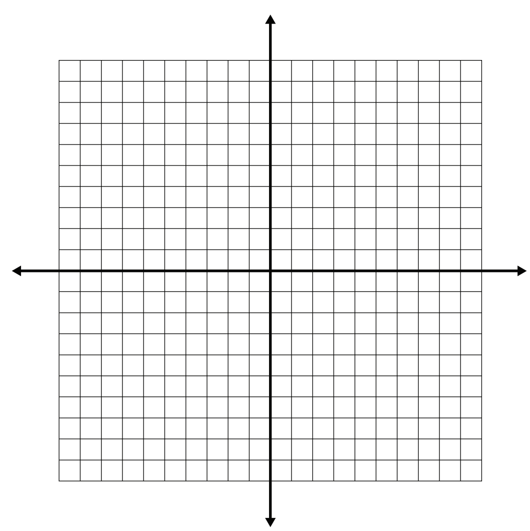
Factor and graph $P(x) = x^3 - 2x^2 - x - 6$

Polynomial Factoring

a) Factor algebraically using the integral zero theorem.

b) Use technology to graph the polynomial. Can the polynomial be factored using just the graph?

c) Can P(x) be factored any other way?



Example 7

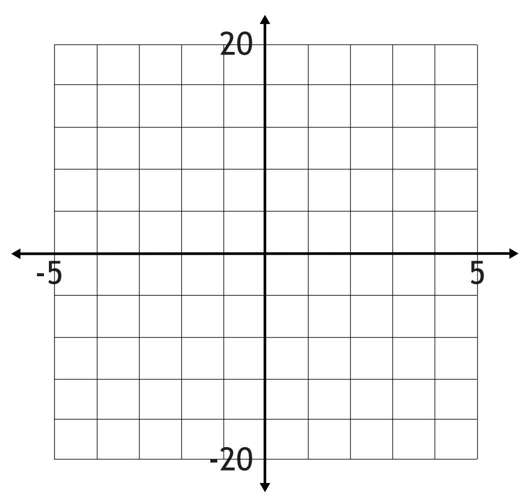
Factor and graph $P(x) = x^4 - 16$

Polynomial Factoring

a) Factor algebraically using the integral zero theorem.

b) Use technology to graph the polynomial. Can the polynomial be factored using just the graph?

c) Can P(x) be factored any other way?



$x^3 - 5x^2 + 2x + 8$

$(x + 1)(x - 2)(x - 4)$

Example 8

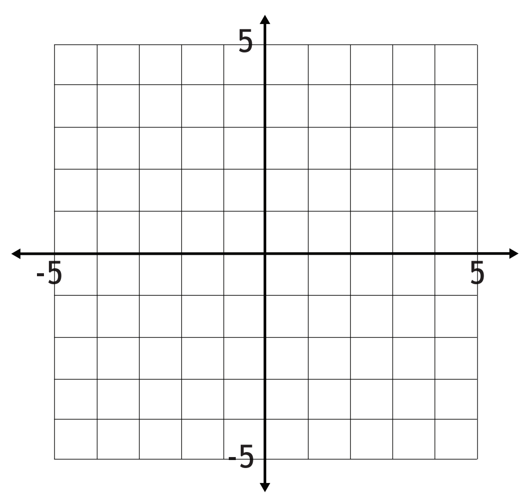
Factor and graph $P(x) = x^5 - 3x^4 - 5x^3 + 27x^2 - 32x + 12$

Polynomial Factoring

a) Factor algebraically using the integral zero theorem.

b) Use technology to graph the polynomial. Can the polynomial be factored using just the graph?

c) Can P(x) be factored any other way?



$x^3 - 5x^2 + 2x + 8$

$(x + 1)(x - 2)(x - 4)$

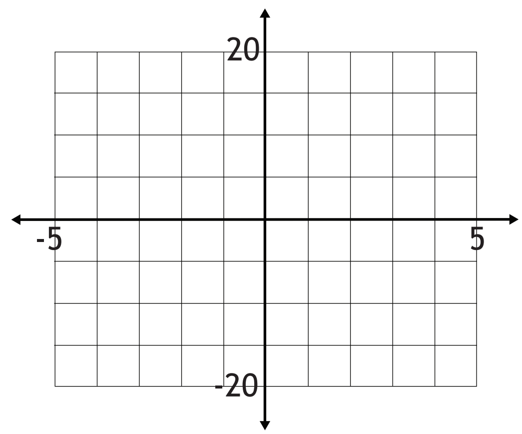
Polynomial, Radical, and Rational Functions
LESSON THREE - *Polynomial Factoring*
Lesson Notes

Example 9

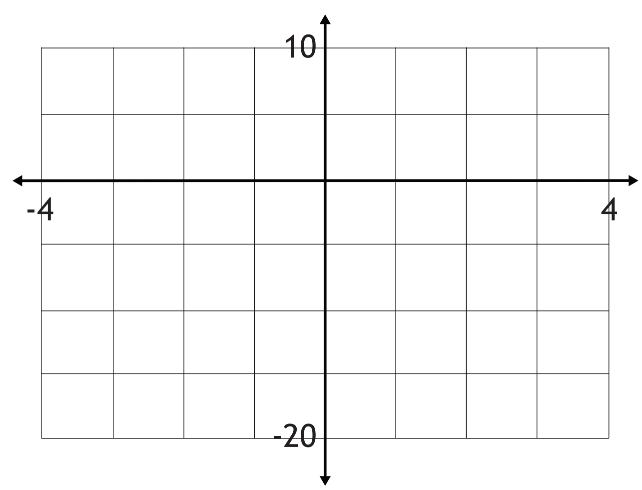
Given the zeros of a polynomial and a point on its graph, find the polynomial function. You may leave the polynomial in factored form. Sketch each graph.

Find the Polynomial Function

a) P(x) has zeros of -4, 0, 0, and 1.
The graph passes through the point (-1, -3).



b) P(x) has zeros of -1, -1, and 2.
The graph passes through the point (1, -8).



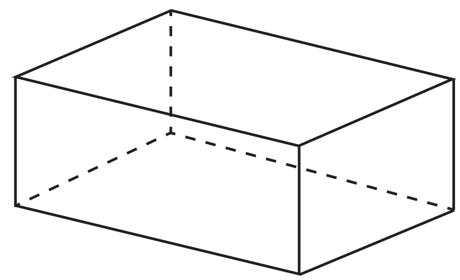
$x^3 - 5x^2 + 2x + 8$

$(x + 1)(x - 2)(x - 4)$

Example 10

Problem Solving

A rectangular prism has a volume of 1050 cm^3. If the height of the prism is 3 cm less than the width of the base, and the length of the base is 5 cm greater than the width of the base, find the dimensions of the rectangular prism. Solve algebraically.



Example 11

Find three consecutive integers with a product of -336. Solve algebraically.

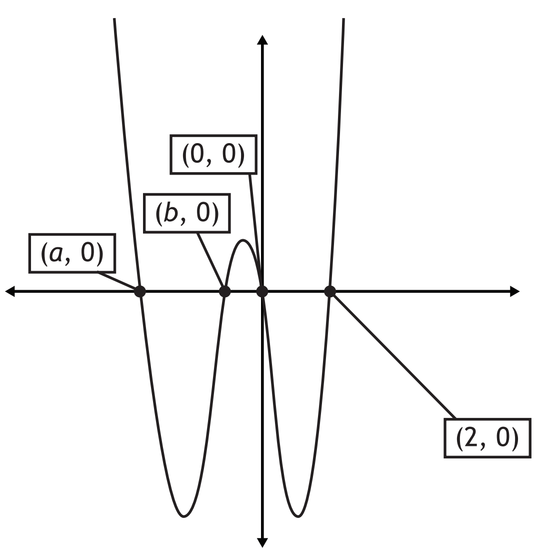
Example 12

If k, $3k$, and $-3k/2$ are zeros of $P(x) = x^3 - 5x^2 - 6kx + 36$, and $k > 0$, find k and write the factored form of the polynomial.

Problem Solving

Example 13

Given the graph of $P(x) = x^4 + 2x^3 - 5x^2 - 6x$ and various points on the graph, determine the values of a and b. Solve algebraically.



$x^3 - 5x^2 + 2x + 8$

$(x + 1)(x - 2)(x - 4)$

Example 14 Solve each equation algebraically. Check with a graphing calculator.

Polynomial Equations

a) $x^3 - 3x^2 - 10x + 24 = 0$

b) $3x^3 + 8x^2 + 4x - 1 = 0$

Quadratic Formula

From Math 20-1:
The roots of a quadratic equation with the form $ax^2 + bx + c = 0$ can be found with the ***quadratic formula***:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1

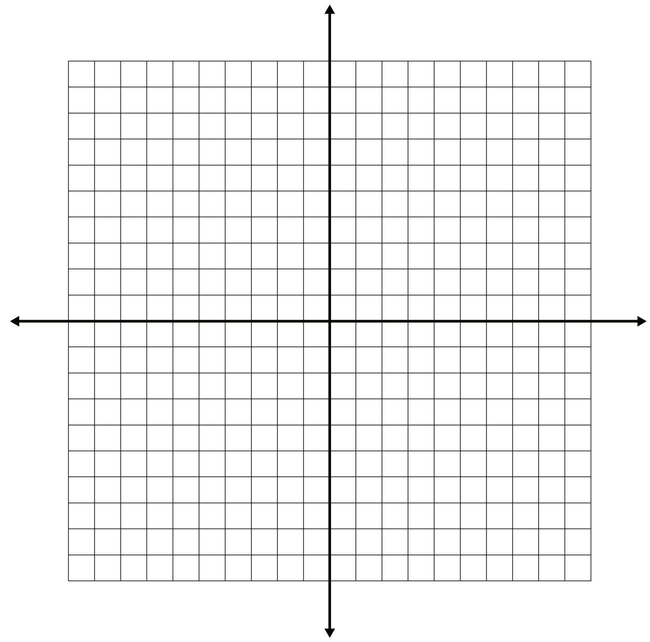
Introduction to Radical Functions

Radical Functions

a) Fill in the table of values for the function $f(x) = \sqrt{x}$

x	*f(x)*
-1	
0	
1	
4	
9	

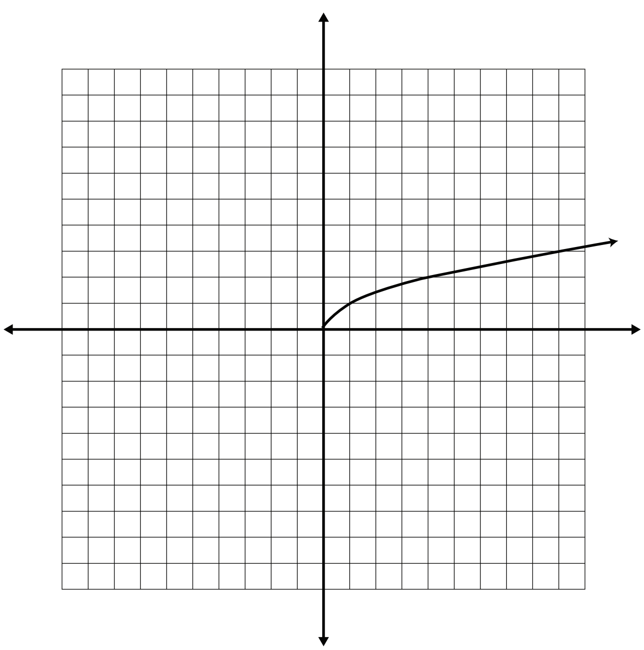
b) Draw the graph of the function $f(x) = \sqrt{x}$ and state the domain and range.



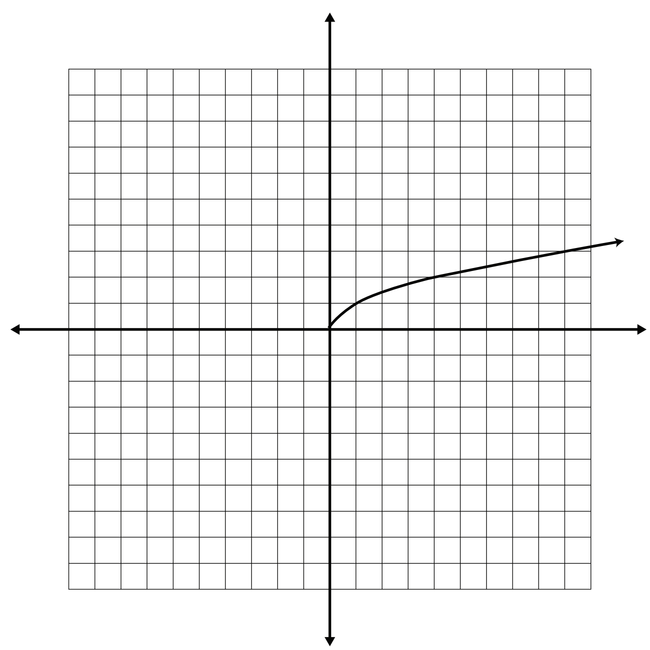
Example 2

Graph each function. The graph of $y = \sqrt{x}$ is provided as a reference.

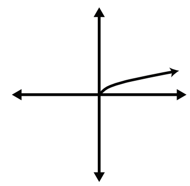
a) $f(x) = -\sqrt{x}$ ***reflection about the x-axis***



b) $f(x) = \sqrt{-x}$ ***reflection about the y-axis***



Polynomial, Radical, and Rational Functions
LESSON FOUR - *Radical Functions*
Lesson Notes

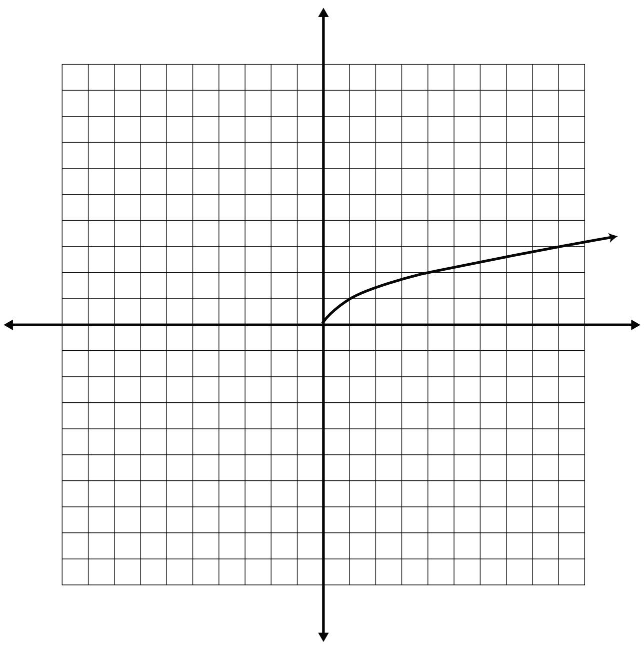


$y = \sqrt{x}$

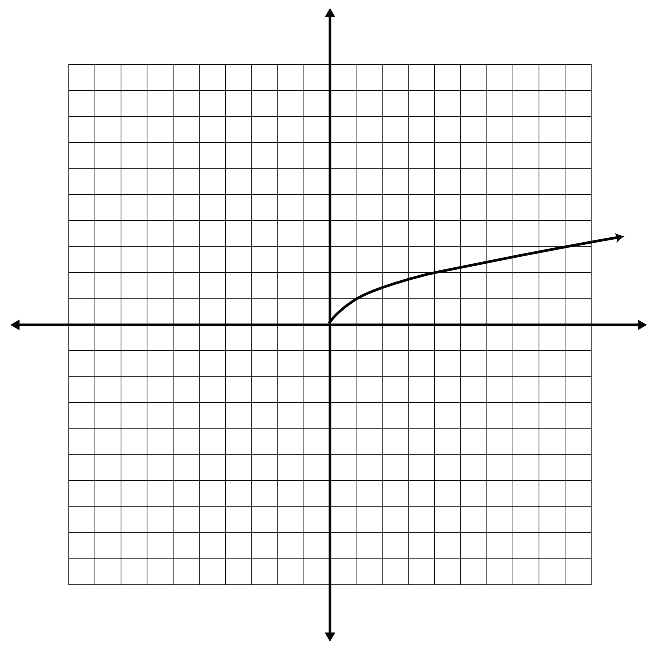
Example 3 Graph each function.
The graph of $y = \sqrt{x}$ is provided as a reference.

Transformations of Radical Functions

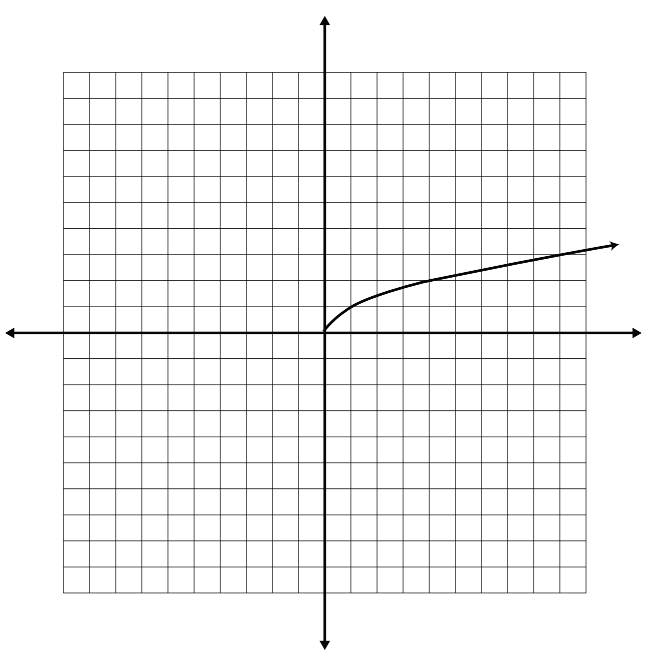
a) $f(x) = 2\sqrt{x}$ ***vertical stretch (double)***



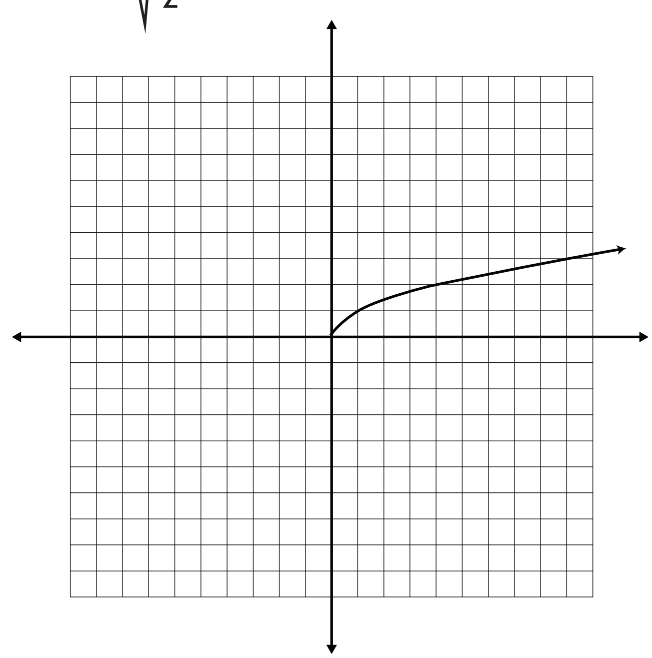
b) $f(x) = \frac{1}{2}\sqrt{x}$ ***vertical stretch (half)***



c) $f(x) = \sqrt{2x}$ ***horizontal stretch (half)***



d) $f(x) = \sqrt{\frac{1}{2}x}$ ***horizontal stretch (double)***



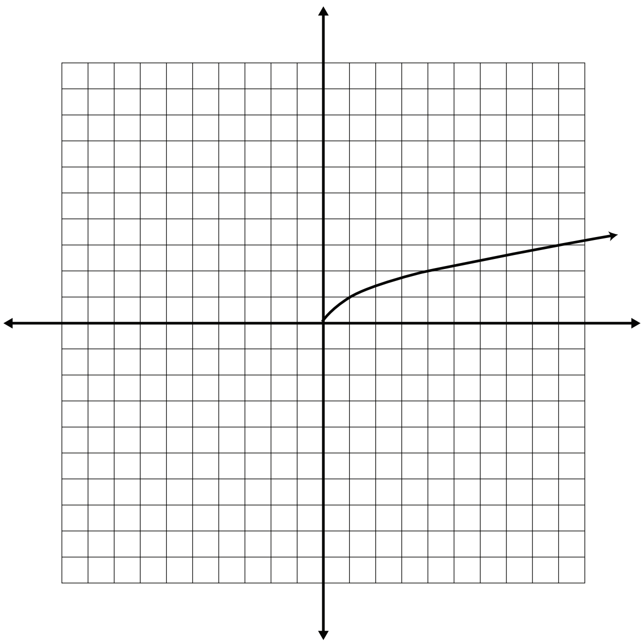
$y = \sqrt{x}$

Example 4

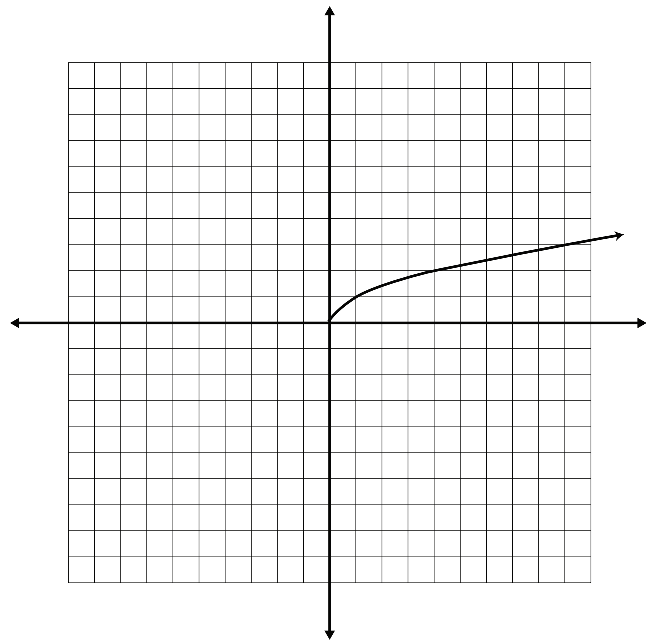
Graph each function.
The graph of $y = \sqrt{x}$ is provided as a reference.

Transformations of Radical Functions

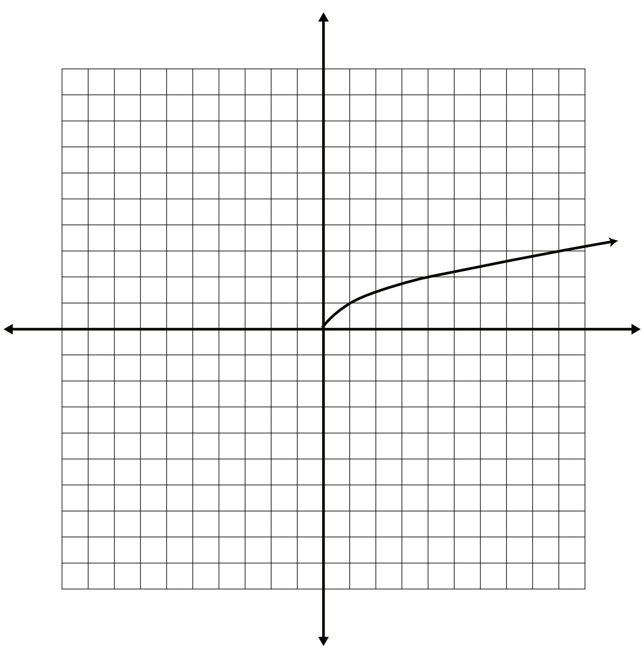
a) $f(x) = \sqrt{x} - 5$ ***vertical translation (down)***



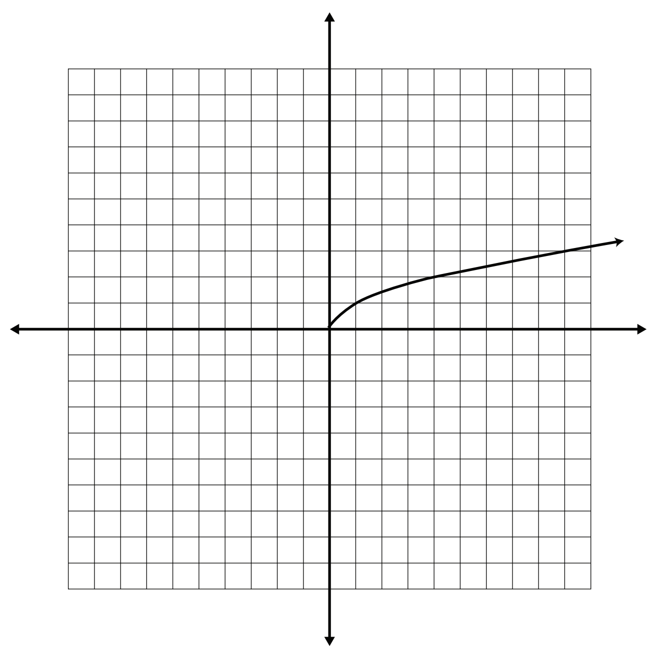
b) $f(x) = \sqrt{x} + 2$ ***vertical translation (up)***

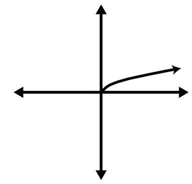


c) $f(x) = \sqrt{x - 1}$ ***horizontal translation (right)***



d) $f(x) = \sqrt{x + 7}$ ***horizontal translation (left)***



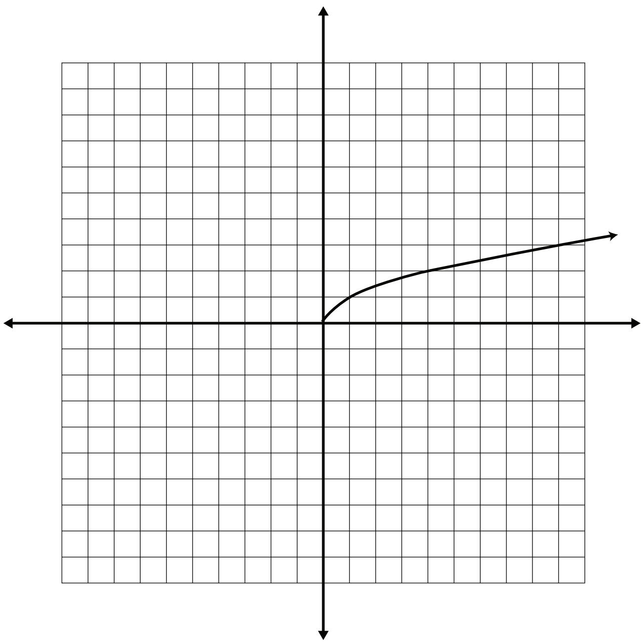


$y = \sqrt{x}$

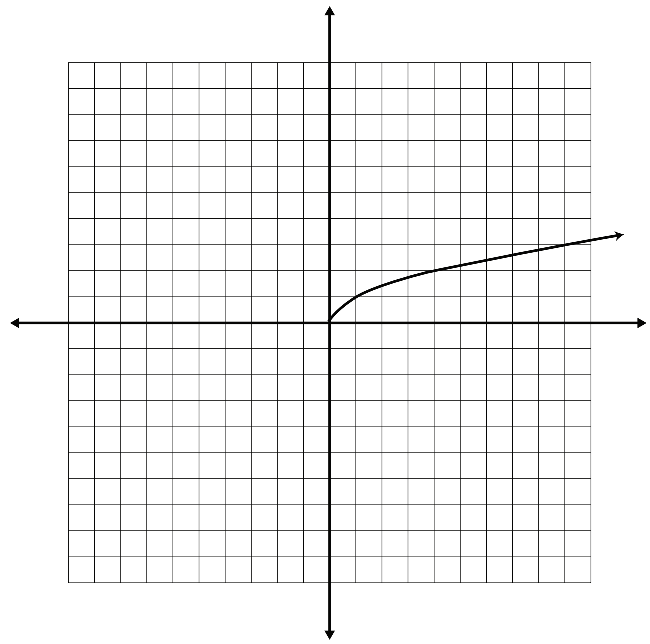
Example 5 Graph each function.
The graph of $y = \sqrt{x}$ is provided as a reference.

Transformations of Radical Functions

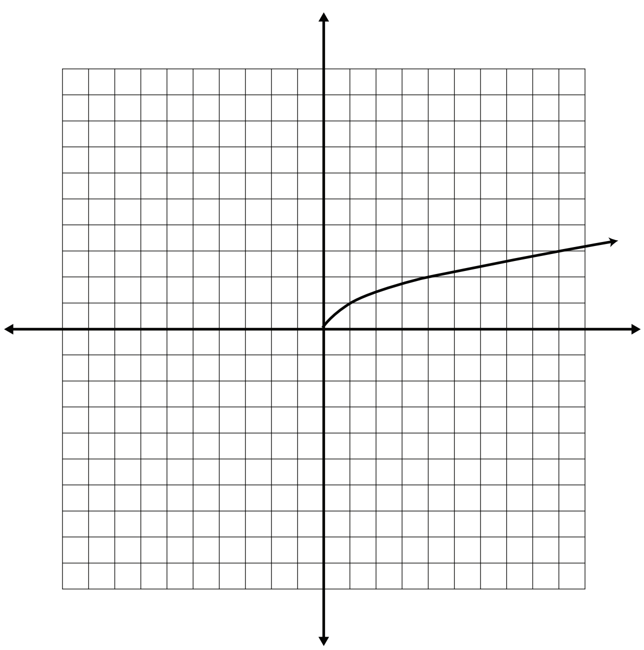
a) $f(x) = \sqrt{x - 3} + 2$



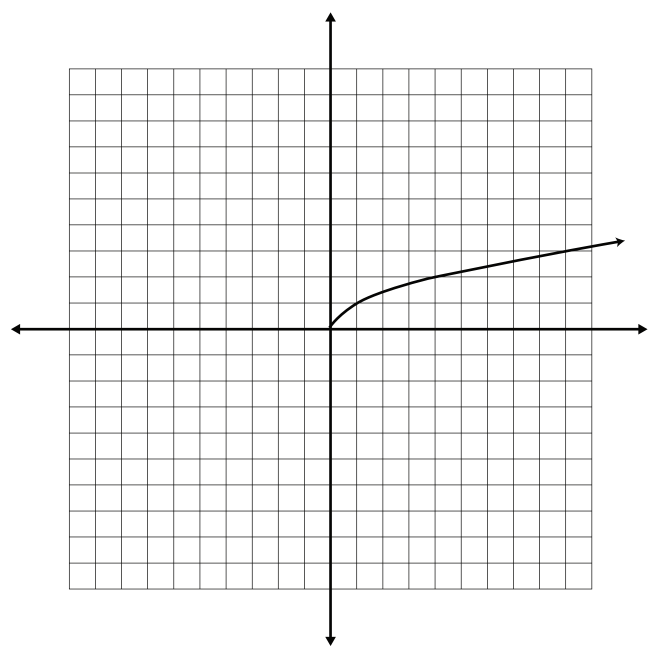
b) $f(x) = 2\sqrt{x + 4}$

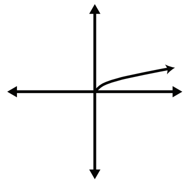


c) $f(x) = -\sqrt{x} - 3$



d) $f(x) = \sqrt{-2x - 4}$



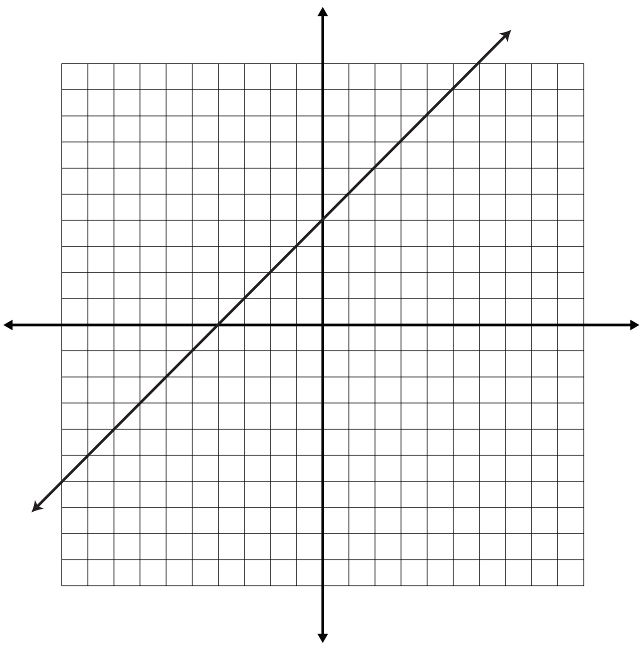


Example 6

Given the graph of $y = f(x)$,
graph $y = \sqrt{f(x)}$ on the same grid.

Square Root of an Existing Function

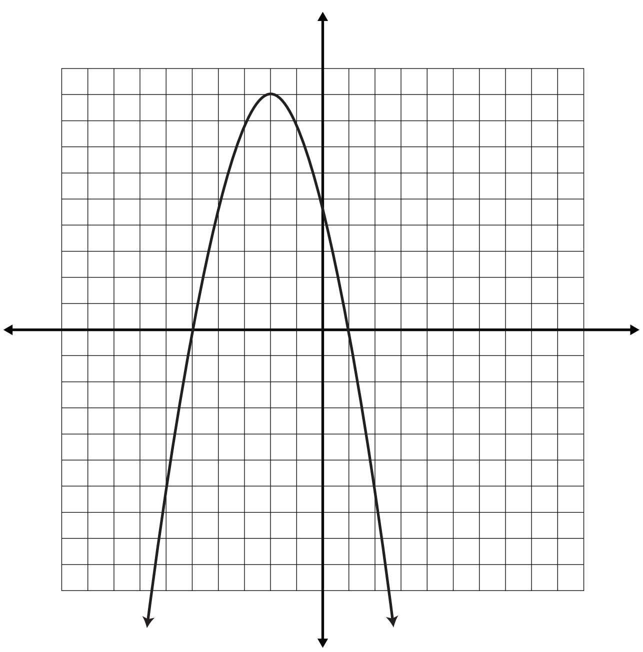
a) $y = x + 4$



Domain & Range
for $y = f(x)$

Domain & Range
for $y = \sqrt{f(x)}$

b) $y = -(x + 2)^2 + 9$



Domain & Range
for $y = f(x)$

Domain & Range
for $y = \sqrt{f(x)}$

Set-Builder Notation

A ***set*** is simply a collection of numbers, such as {1, 4, 5}. We use ***set-builder notation*** to outline the rules governing members of a set.

$\{x \mid x \in R, x \geq -1\}$

State the variable. List conditions on the variable.

In words: "The variable is x, such that x can be any real number with the condition that $x \geq -1$".

As a shortcut, set-builder notation can be reduced to just the most important condition.

$x \geq -1$

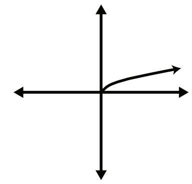
While this resource uses the shortcut for brevity, as set-builder notation is covered in previous courses, Math 30-1 students *are* expected to know how to read and write full set-builder notation.

Interval Notation

Math 30-1 students are expected to know that domain and range can be expressed using ***interval notation***.

() - Round Brackets: Exclude point from interval.
[] - Square Brackets: Include point in interval.
Infinity ∞ always gets a round bracket.

Examples: $x \geq -5$ becomes $[-5, \infty)$;
$1 < x \leq 4$ becomes $(1, 4]$;
$x \in R$ becomes $(-\infty, \infty)$;
$-8 \leq x < 2$ or $5 \leq x < 11$ becomes $[-8, 2) \cup [5, 11)$,
where U means "or", or ***union of sets***;
$x \in R, x \neq 2$ becomes $(-\infty, 2) \cup (2, \infty)$;
$-1 \leq x \leq 3, x \neq 0$ becomes $[-1, 0) \cup (0, 3]$.

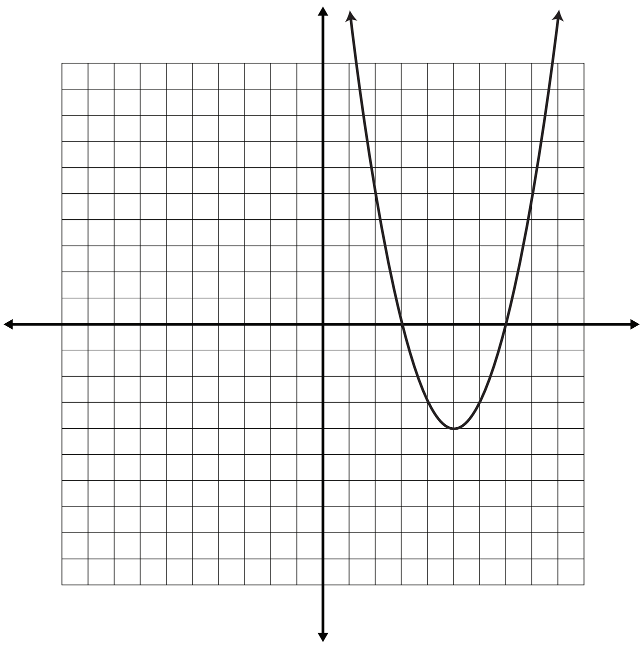


Example 7

Given the graph of $y = f(x)$, graph $y = \sqrt{f(x)}$ on the same grid.

Square Root of an Existing Function

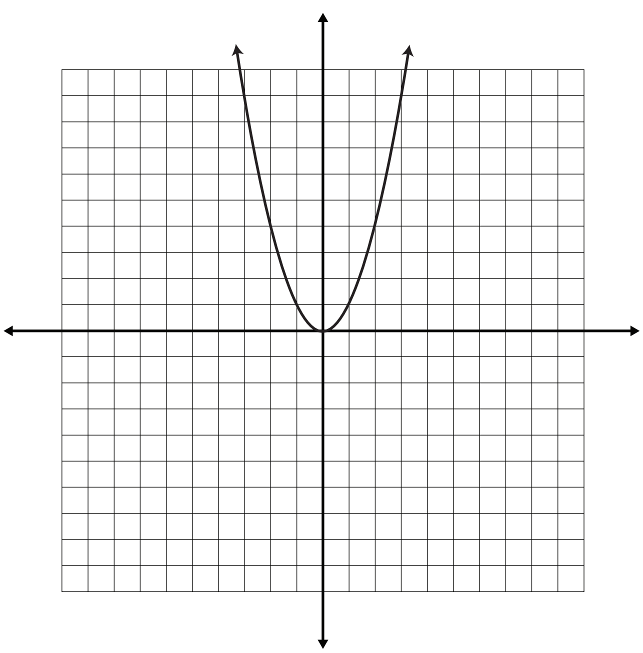
a) $y = (x - 5)^2 - 4$



Domain & Range for $y = f(x)$

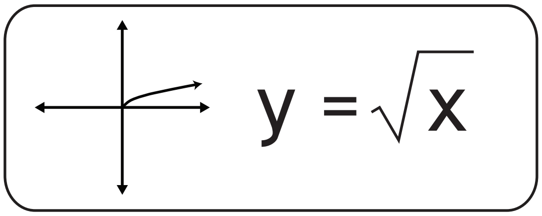
Domain & Range for $y = \sqrt{f(x)}$

b) $y = x^2$



Domain & Range for $y = f(x)$

Domain & Range for $y = \sqrt{f(x)}$



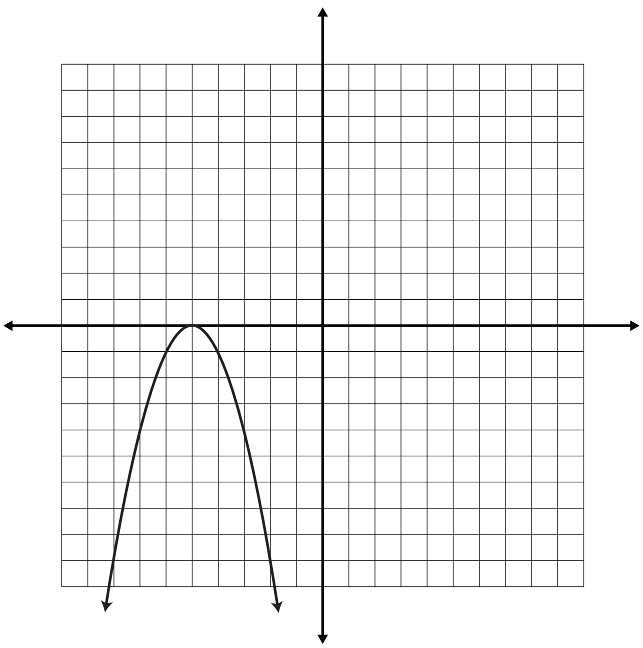
Example 8

Given the graph of $y = f(x)$,

graph $y = \sqrt{f(x)}$ on the same grid.

Square Root of an Existing Function

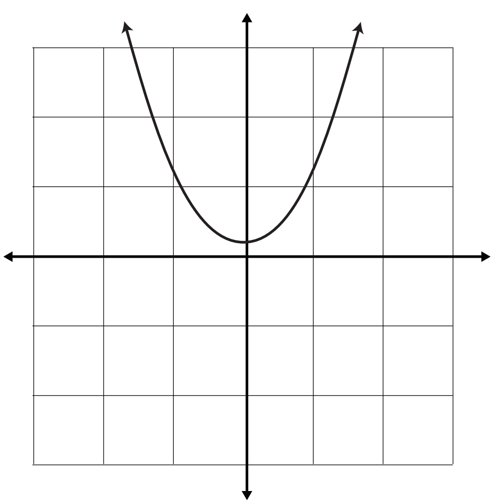
a) $y = -(x + 5)^2$



Domain & Range for $y = f(x)$

Domain & Range for $y = \sqrt{f(x)}$

b) $y = x^2 + 0.25$



Domain & Range for $y = f(x)$

Domain & Range for $y = \sqrt{f(x)}$

Lesson Notes

$y = \sqrt{x}$

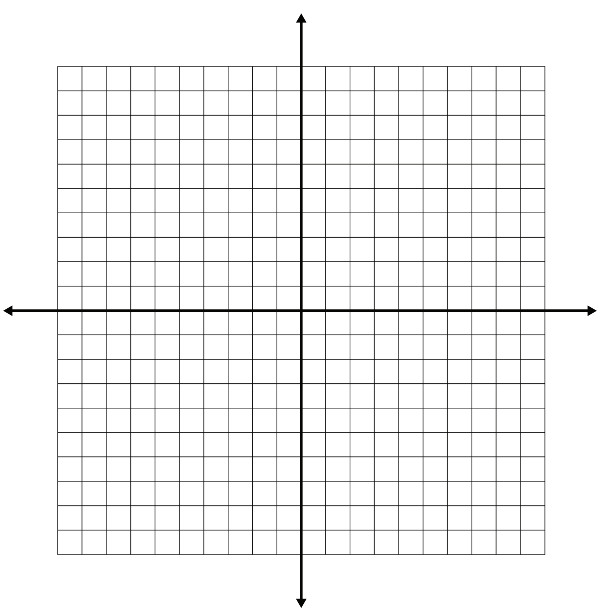
Example 9

Solve the radical equation $\sqrt{x + 2} = 3$ in three different ways.

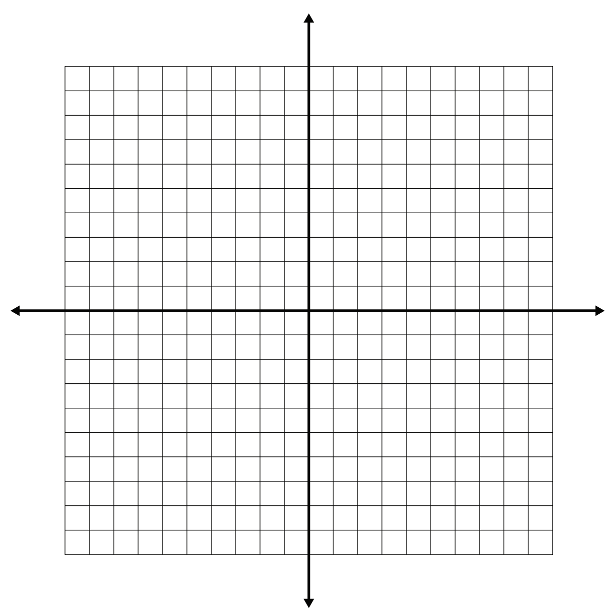
Radical Equations

a) Solve algebraically and check for extraneous roots.

b) Solve by finding the point of intersection of a system of equations.



c) Solve by finding the x-intercept(s) of a single function.



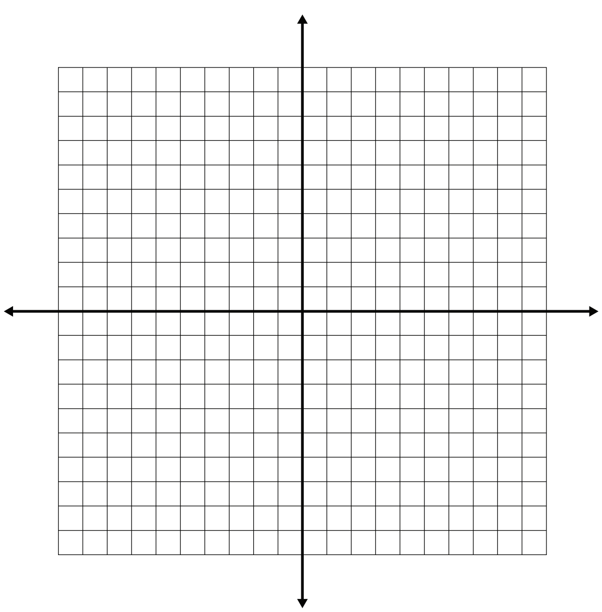
Example 10

Solve the radical equation $x = \sqrt{x + 2}$ in three different ways.

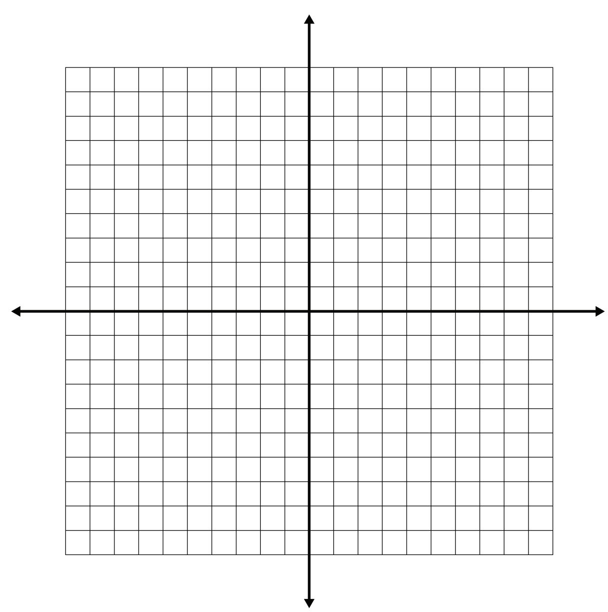
Radical Equations

a) Solve algebraically and check for extraneous roots.

b) Solve by finding the point of intersection of a system of equations.



c) Solve by finding the x-intercept(s) of a single function.



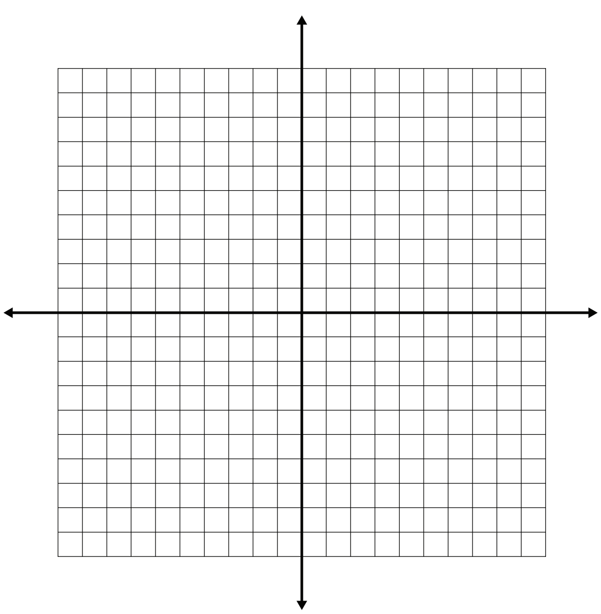
Example 11

Solve the radical equation $2\sqrt{x + 3} = x + 3$ in three different ways.

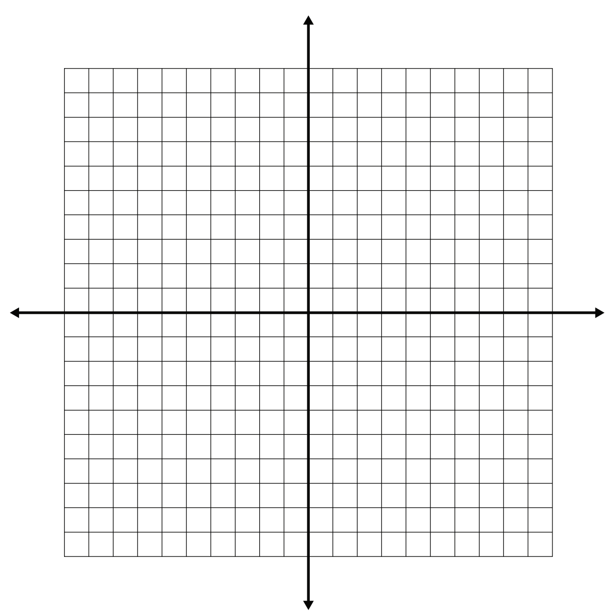
Radical Equations

a) Solve algebraically and check for extraneous roots.

b) Solve by finding the point of intersection of a system of equations.



c) Solve by finding the x-intercept(s) of a single function.

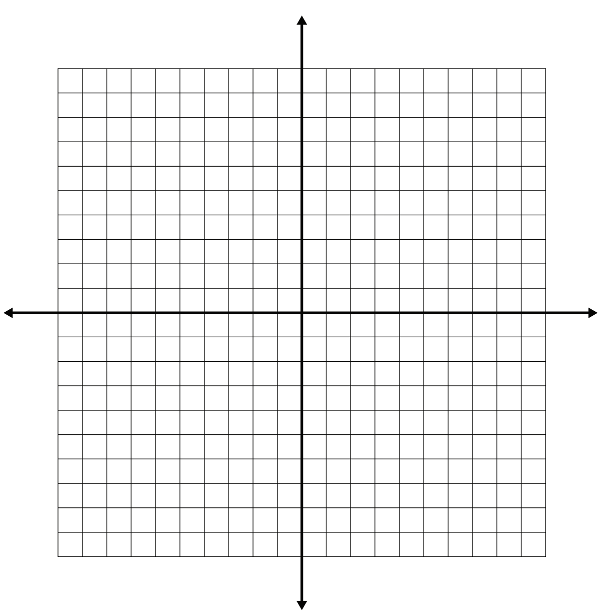


Example 12 Solve the radical equation $\sqrt{16 - x^2} = 5$ in three different ways.

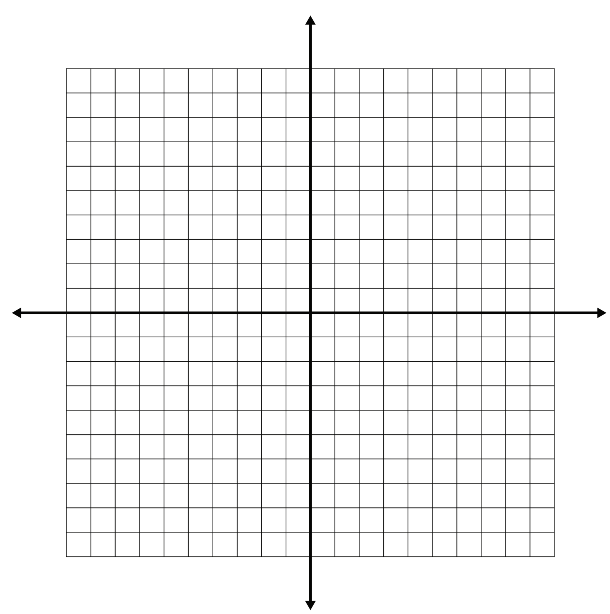
Radical Equations

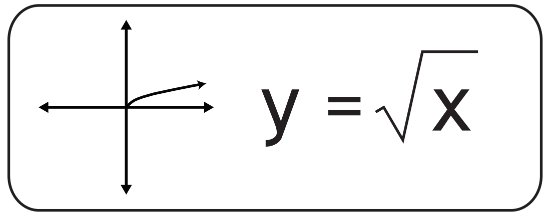
a) Solve algebraically and check for extraneous roots.

b) Solve by finding the point of intersection of a system of equations.



c) Solve by finding the x-intercept(s) of a single function.

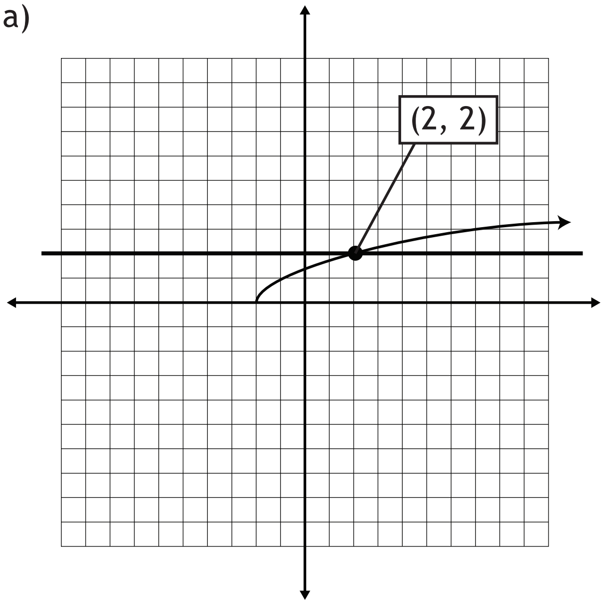




Example 13 Write an equation that can be used to find the point of intersection for each pair of graphs.

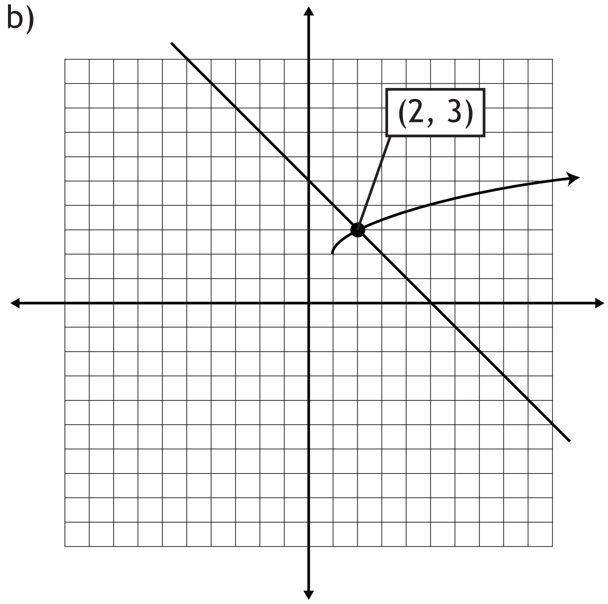
Find the Radical Equation

a)



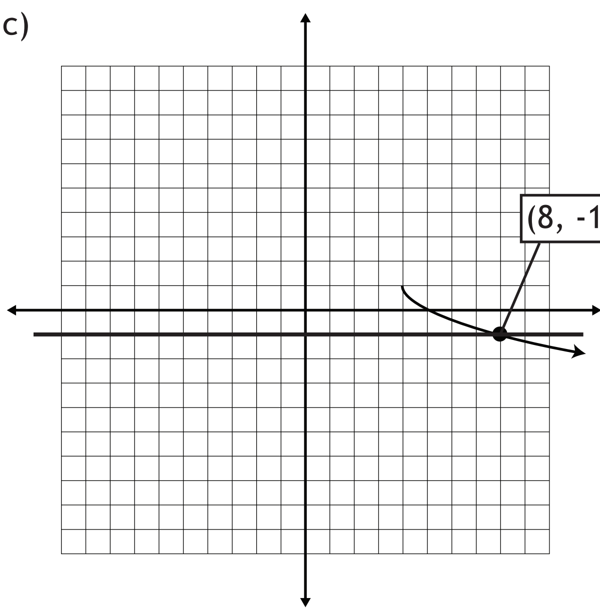
Equation:

b)



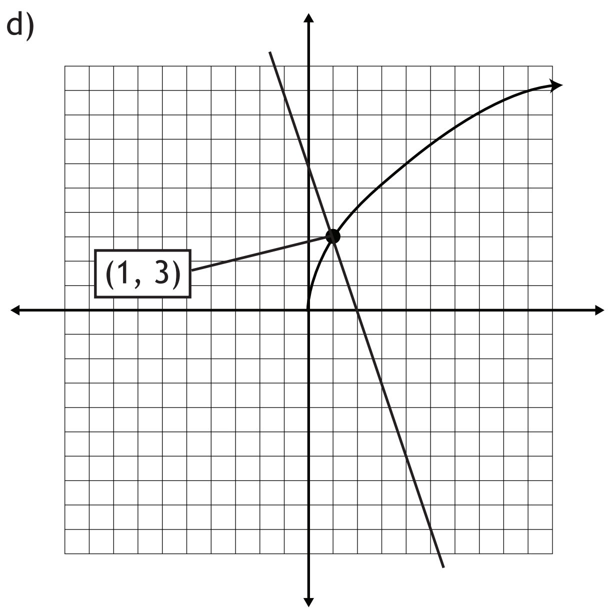
Equation:

c)



Equation:

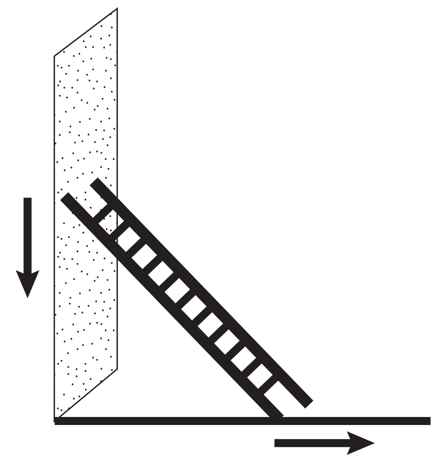
d)



Equation:

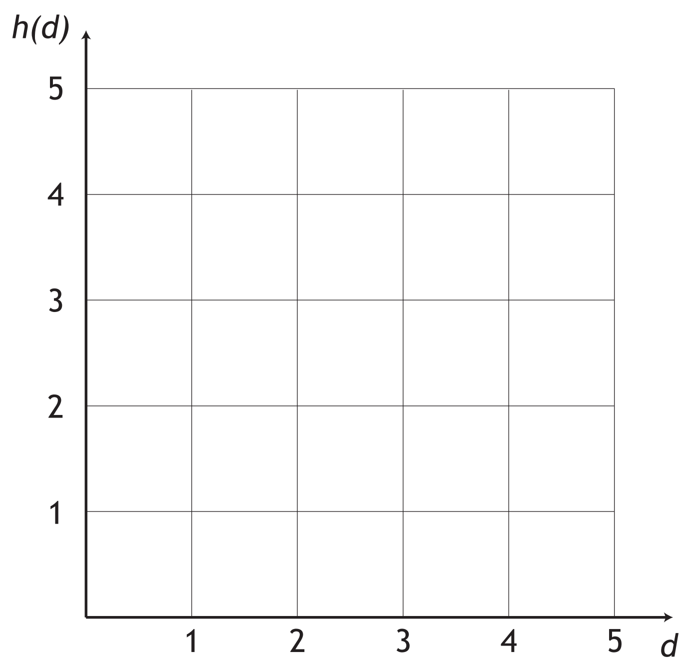
Example 14

A ladder that is 3 m long is leaning against a wall. The base of the ladder is *d* metres from the wall, and the top of the ladder is *h* metres above the ground.

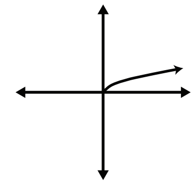


a) Write a function, *h(d)*, to represent the height of the ladder as a function of its base distance *d*.

b) Graph the function and state the domain and range. Describe the ladder's orientation when d = 0 and d = 3.

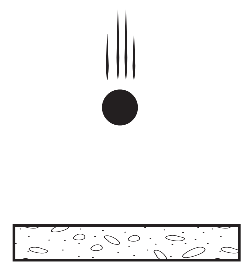


c) How far is the base of the ladder from the wall when the top of the ladder is $\sqrt{5}$ metres above the ground?



$y = \sqrt{x}$

Example 15



If a ball at a height of *h* metres is dropped, the length of time it takes to hit the ground is:

$$\boxed{t = \sqrt{\frac{h}{4.9}}}$$

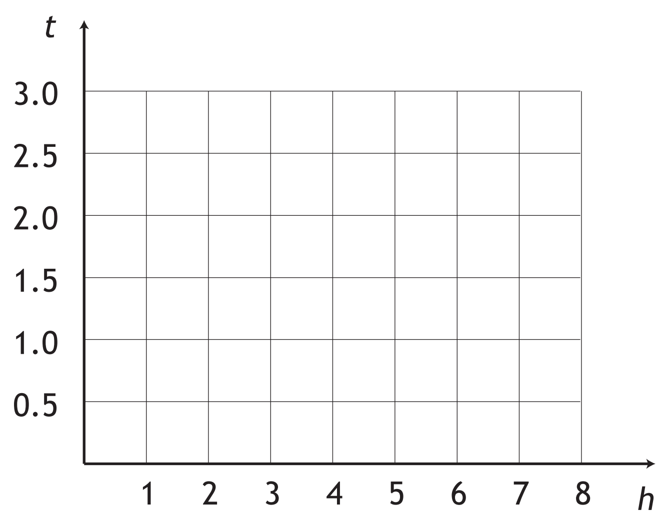
where *t* is the time in seconds.

a) If a ball is dropped from twice its original height, how will that change the time it takes to fall?

b) If a ball is dropped from one-quarter of its original height, how will that change the time it takes to fall?

c) The original height of the ball is 4 m. Complete the table of values and draw the graph. Do your results match the predictions made in parts (a & b)?

h *metres*	***t*** *seconds*
1 *quarter*	
4 *original*	
8 *double*	

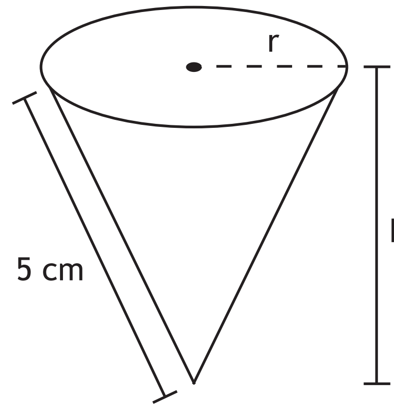


Example 16

A disposable paper cup has the shape of a cone. The volume of the cone is V (cm^3), the radius is r (cm), the height is h (cm), and the slant height is 5 cm.

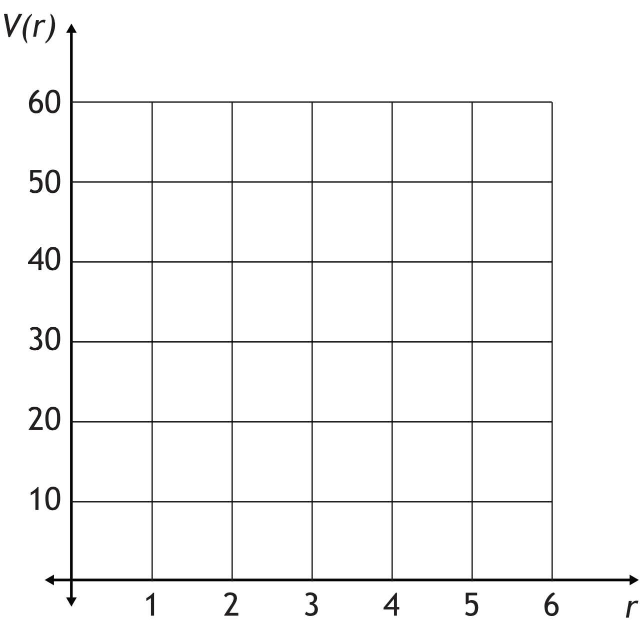
Cone Volume

$$V = \frac{1}{3}\pi r^2 h$$



a) Derive a function, $V(r)$, that expresses the volume of the paper cup as a function of r.

b) Graph the function from part (a) and explain the shape of the graph.



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Lesson Notes

Example 1

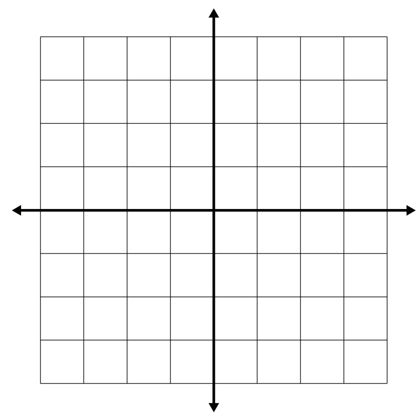
Reciprocal of a ***Linear Function***.

Reciprocal of a Linear Function

a) Fill in the table of values for the function $y = \frac{1}{x}$.

x	*y*
-2	
-1	
-0.5	
-0.25	
0	
0.25	
0.5	
1	
2	

b) Draw the graph of the function $y = \frac{1}{x}$. State the domain and range.



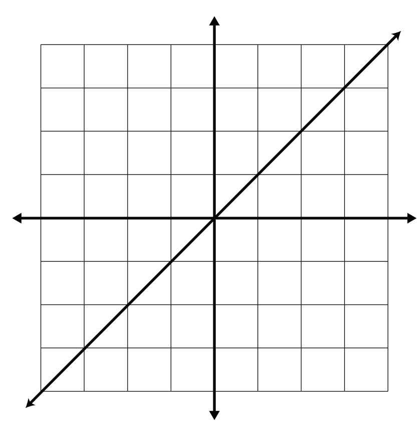
c) Draw the graph of y = x in the same grid used for part (b).
Compare the graph of y = x to the graph of $y = \frac{1}{x}$.

d) Outline a series of steps that can be used to draw the graph of $y = \frac{1}{x}$, starting from y = x.

Step One:

Step Two:

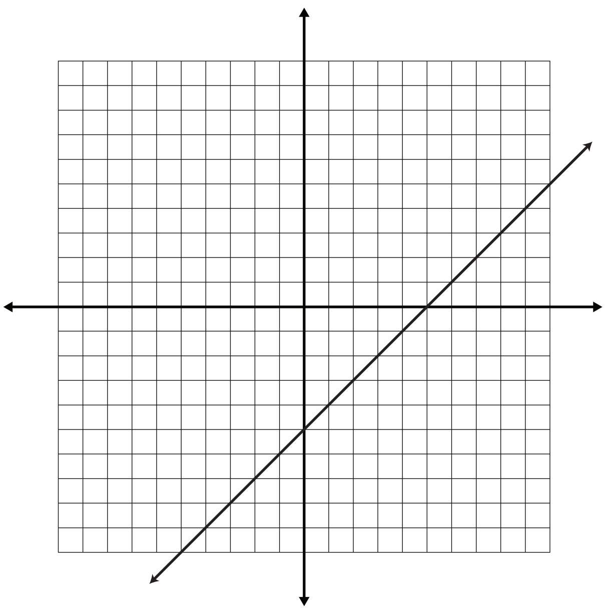
Step Three:



Example 2 Given the graph of $y = f(x)$, draw the graph of $y = \frac{1}{f(x)}$.

Reciprocal of a Linear Function

a) $y = x - 5$

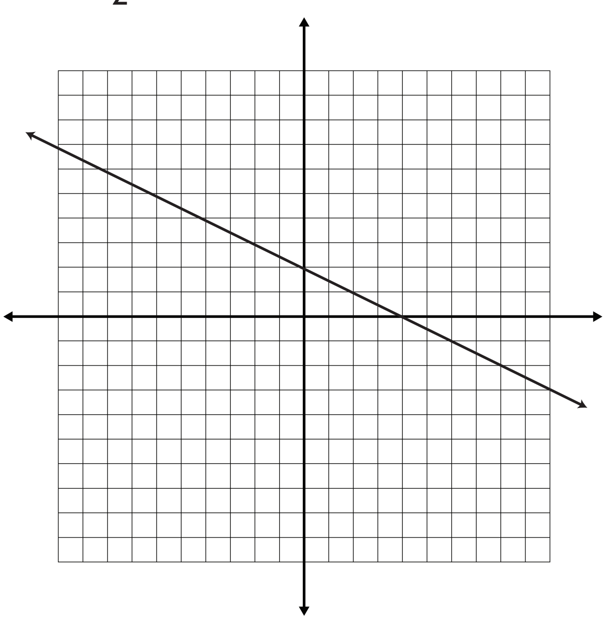


Domain & Range of $y = f(x)$

Domain & Range of $y = \frac{1}{f(x)}$

Asymptote Equations:

b) $y = -\frac{1}{2}x + 2$



Domain & Range of $y = f(x)$

Domain & Range of $y = \frac{1}{f(x)}$

Asymptote Equation(s):

Example 3

Reciprocal of a ***Quadratic Function***.

Reciprocal of a Quadratic Function

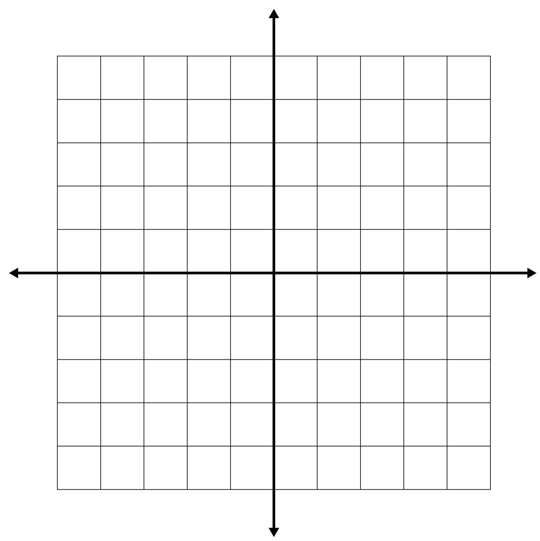
a) Fill in the table of values for the function $y = \frac{1}{x^2 - 4}$.

x	y
-3	
-2	
-1	
0	
1	
2	
3	

x	y
-2.05	
-1.95	

x	y
1.95	
2.05	

b) Draw the graph of the function $y = \frac{1}{x^2 - 4}$. State the domain and range.



c) Draw the graph of $y = x^2 - 4$ in the same grid used for part (b). Compare the graph of $y = x^2 - 4$ to the graph of $y = \frac{1}{x^2 - 4}$.

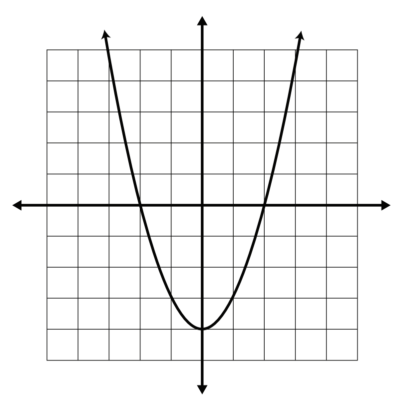
d) Outline a series of steps that can be used to draw the graph of $y = \frac{1}{x^2 - 4}$, starting from $y = x^2 - 4$.

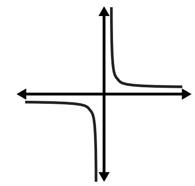
Step One:

Step Two:

Step Three:

Step Four:



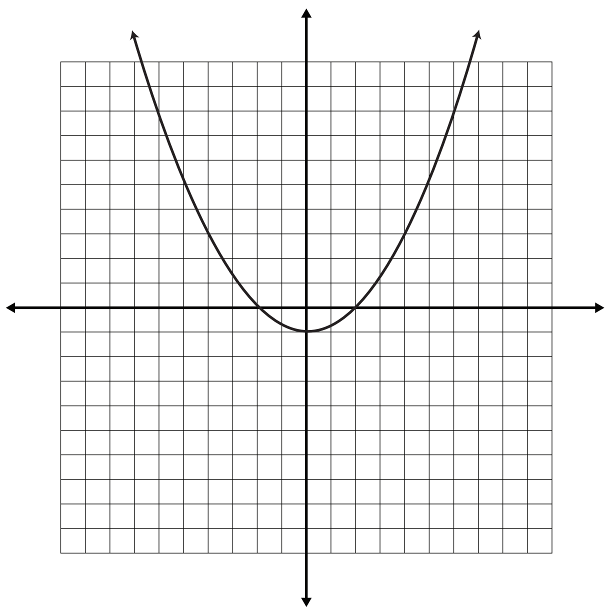


Example 4

Given the graph of $y = f(x)$, draw the graph of $y = \frac{1}{f(x)}$.

Reciprocal of a Quadratic Function

a) $y = \frac{1}{4}x^2 - 1$

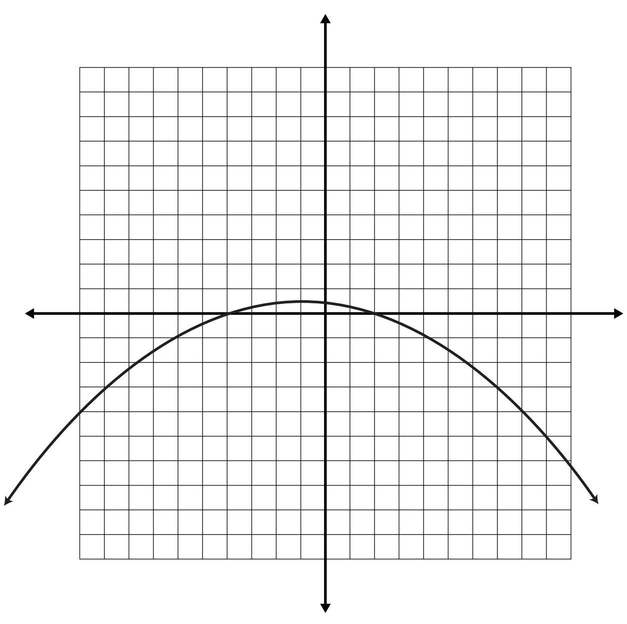


Domain & Range of $y = f(x)$

Domain & Range of $y = \frac{1}{f(x)}$

Asymptote Equation(s):

b) $y = -\frac{1}{18}(x + 1)^2 + \frac{1}{2}$

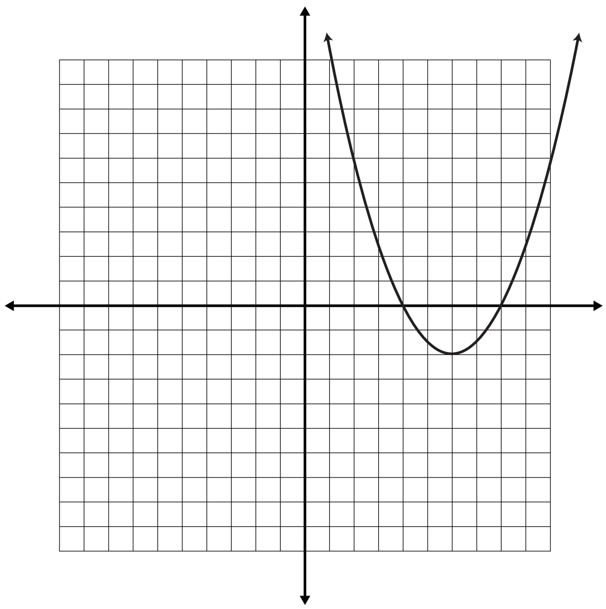


Domain & Range of $y = f(x)$

Domain & Range of $y = \frac{1}{f(x)}$

Asymptote Equation(s):

c) $y = \frac{1}{2}(x - 6)^2 - 2$

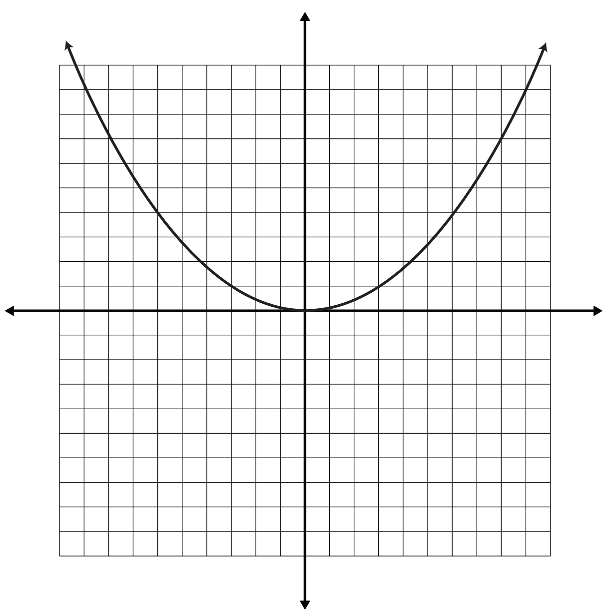


Domain & Range of $y = f(x)$

Domain & Range of $y = \frac{1}{f(x)}$

Asymptote Equation(s):

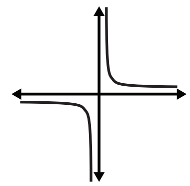
d) $y = \frac{1}{9}x^2$



Domain & Range of $y = f(x)$

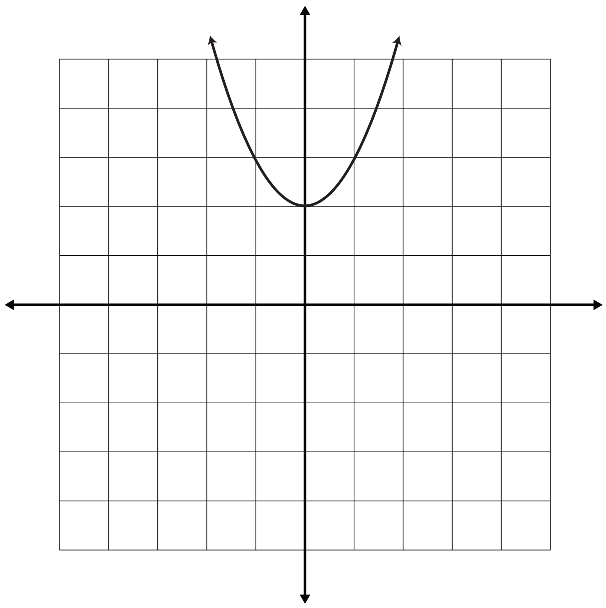
Domain & Range of $y = \frac{1}{f(x)}$

Asymptote Equation(s):



$y = \frac{1}{x}$

e) $y = x^2 + 2$

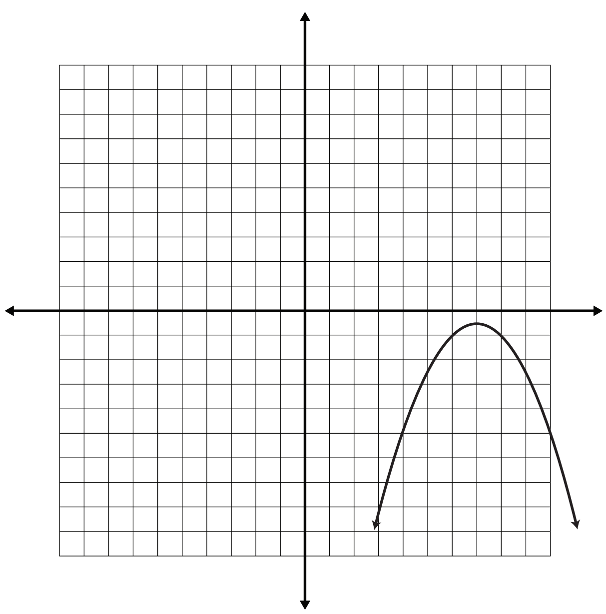


Domain & Range of $y = f(x)$

Domain & Range of $y = \frac{1}{f(x)}$

Asymptote Equation(s):

f) $y = -\frac{1}{2}(x - 7)^2 - \frac{1}{2}$



Domain & Range of $y = f(x)$

Domain & Range of $y = \frac{1}{f(x)}$

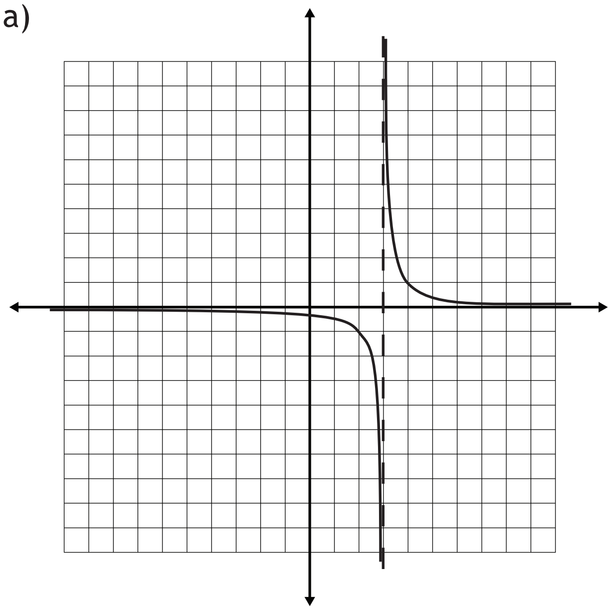
Asymptote Equation(s):

Example 5

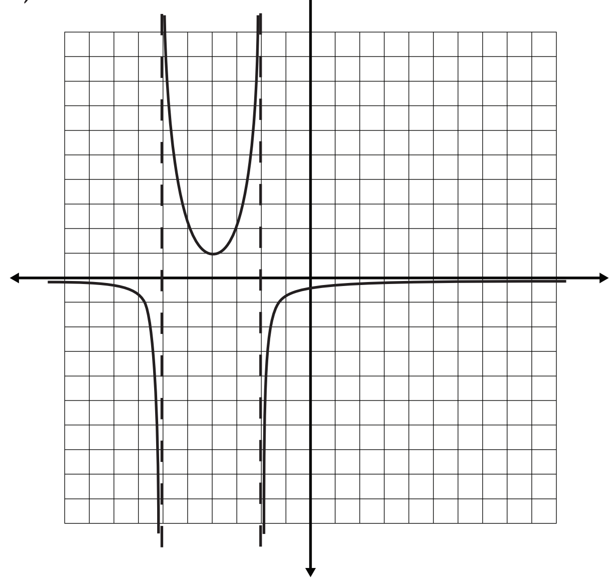
Given the graph of $y = \frac{1}{f(x)}$, draw the graph of $y = f(x)$.

Find the Original Function

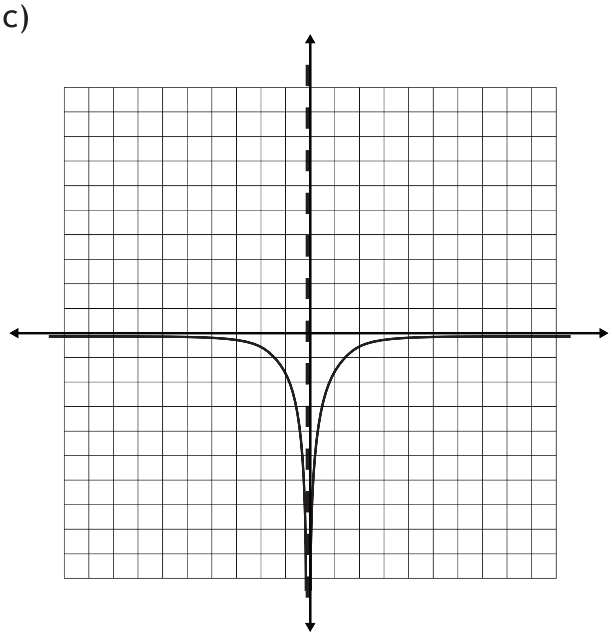
a)



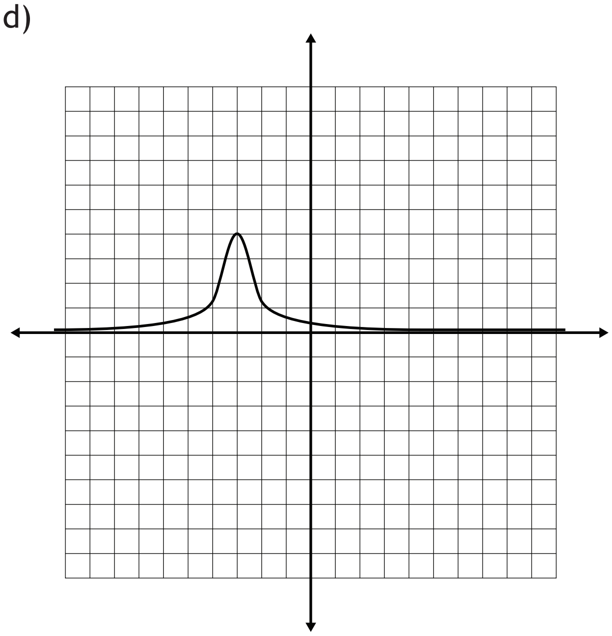
b)



c)



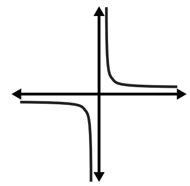
d)



Polynomial, Radical, and Rational Functions

LESSON FIVE - *Rational Functions I*

Lesson Notes



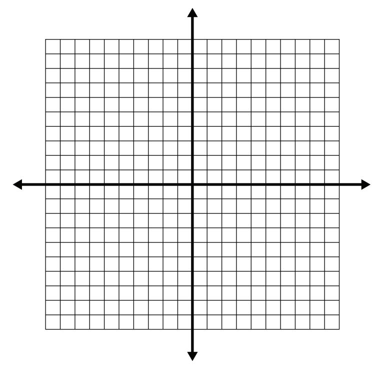
$$y = \frac{1}{x}$$

Example 6

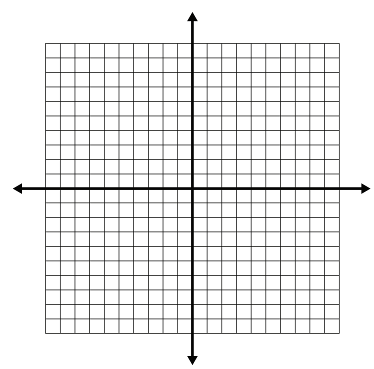
For each function, determine the equations of all asymptotes. Check with a graphing calculator.

Asymptote Equations

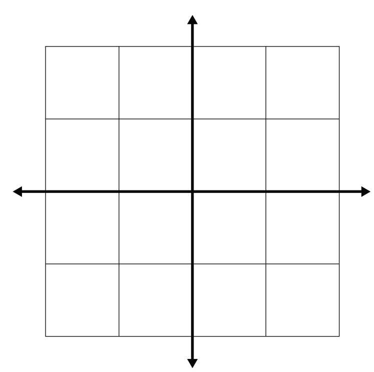
a) $f(x) = \dfrac{1}{2x - 3}$



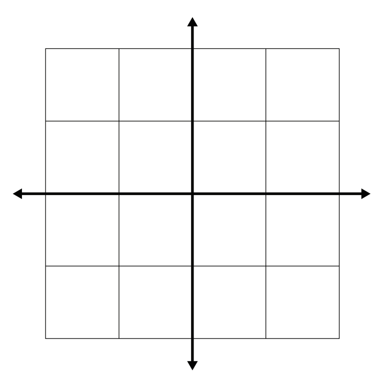
b) $f(x) = \dfrac{1}{x^2 - 2x - 24}$



c) $f(x) = \dfrac{1}{6x^3 - 5x^2 - 4x}$



d) $f(x) = \dfrac{1}{4x^2 + 9}$



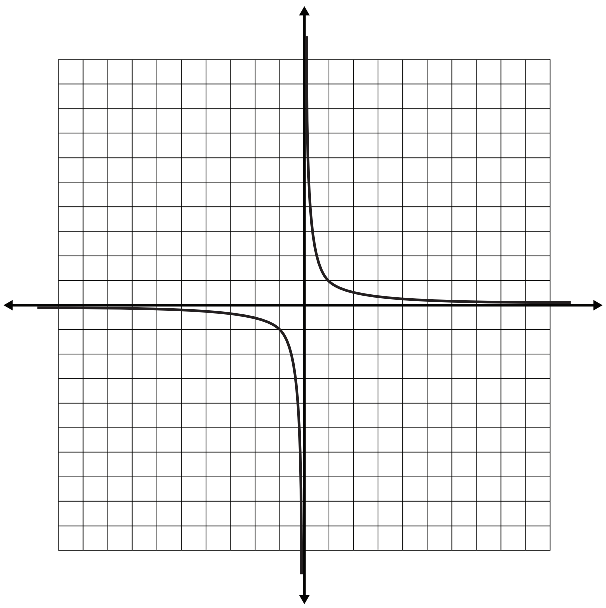
Example 7

Compare each of the following functions to $y = 1/x$ by identifying any stretches or translations, then draw the graph without using technology.

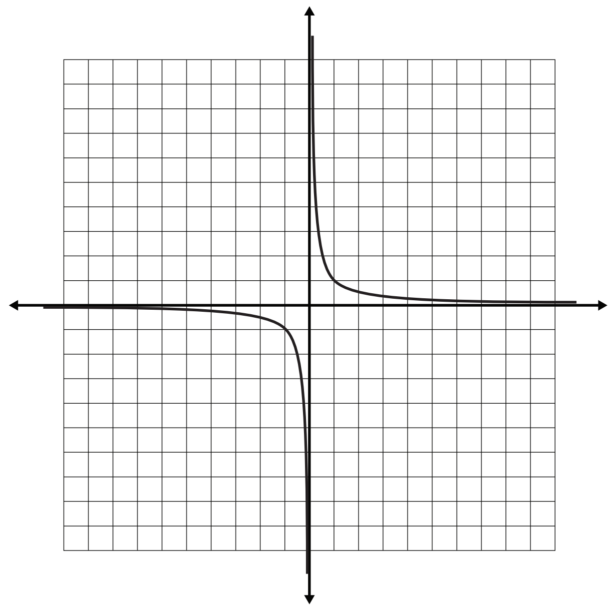
Transformations of Reciprocal Functions

The graph of $y = 1/x$ is provided as a convenience.

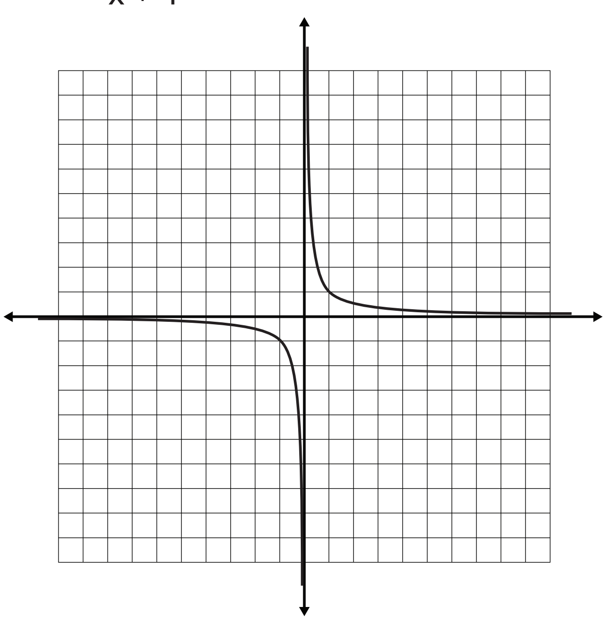
a) $y = \frac{4}{x}$



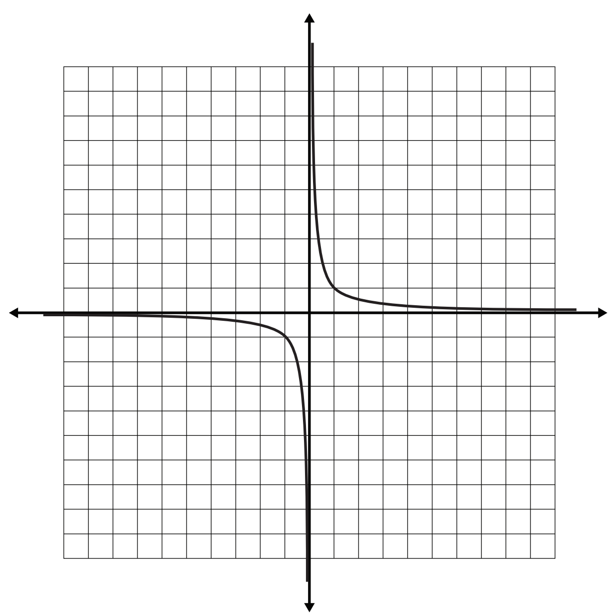
b) $y = \frac{1}{x} - 3$

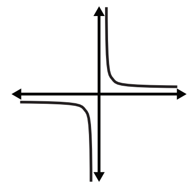


c) $y = \frac{3}{x + 4}$



d) $y = \frac{2}{x - 3} + 2$





$$y = \frac{1}{x}$$

Example 8

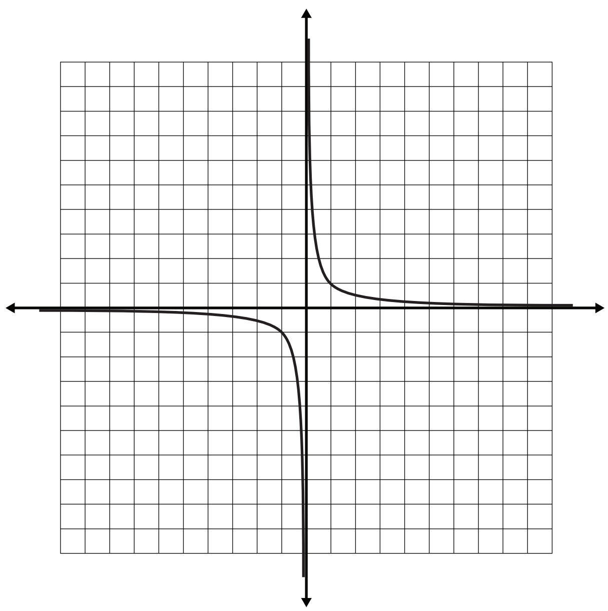
Convert each of the following functions to the form $y = a\left(\frac{1}{x-h}\right) + k$.

Identify the stretches and translations, then draw the graph without using technology.

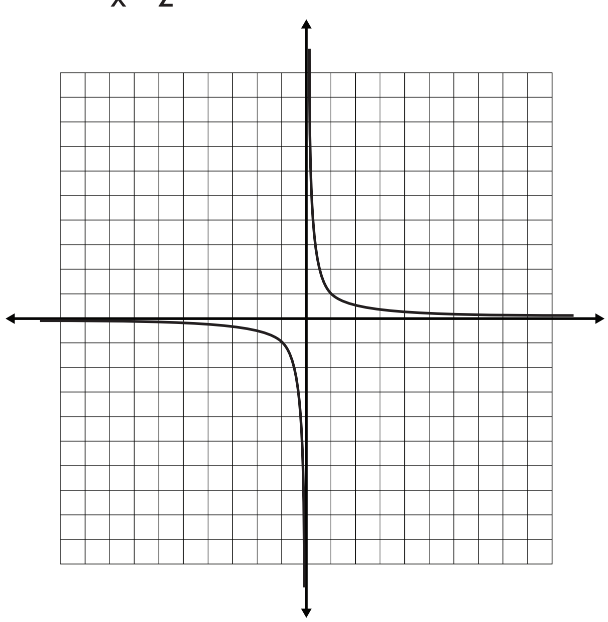
Transformations of Reciprocal Functions

The graph of y = 1/x is provided as a convenience.

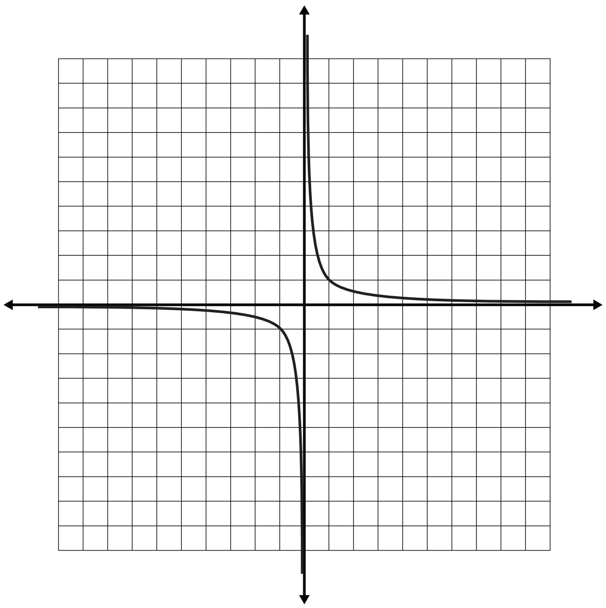
a) $y = \frac{1 - 2x}{x}$



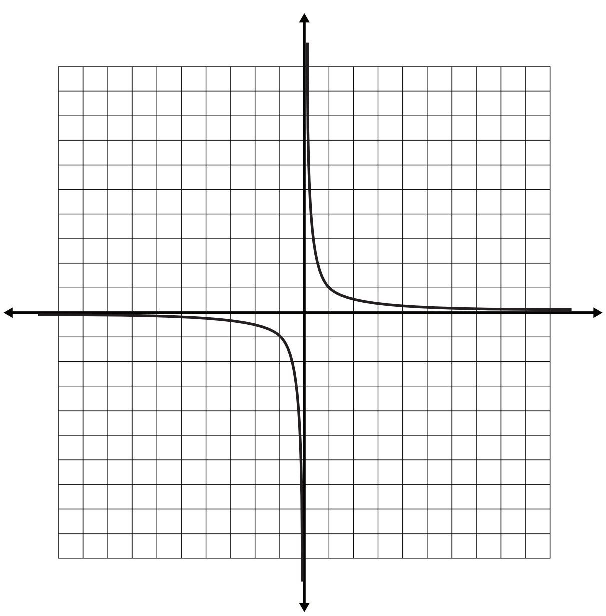
b) $y = \frac{x - 1}{x - 2}$



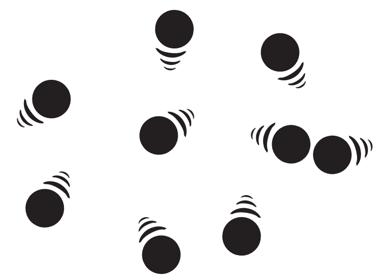
c) $y = \frac{6 - 2x}{x - 1}$



d) $y = \frac{33 - 6x}{x - 5}$



Example 9 ***Chemistry Application: Ideal Gas Law***



The ideal gas law relates the pressure, volume, temperature, and molar amount of a gas with the formula:

$$\boxed{PV = nRT}$$

where P is the pressure in kilopascals (kPa), V is the volume in litres (L), n is the molar amount of the gas (mol), R is the universal gas constant, and T is the temperature in kelvins (K).

An ideal gas law experiment uses 0.011 mol of a gas at a temperature of 273.15 K.

a) If the temperature and molar amount of the gas are held constant, the ideal gas law follows a reciprocal relationship and can be written as a rational function, P(V). Write this function.

b) If the original volume of the gas is doubled, how will the pressure change?

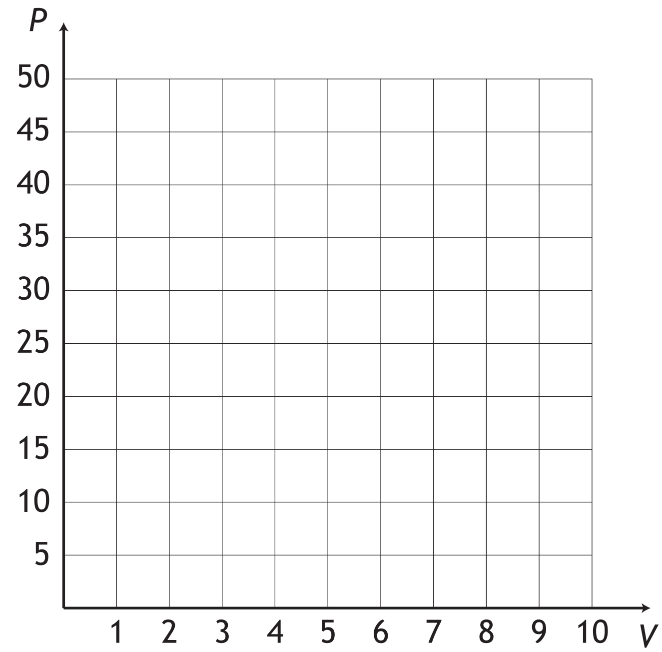
c) If the original volume of the gas is halved, how will the pressure change?

d) If P(5.0 L) = 5.0 kPa, determine the experimental value of the universal gas constant R.

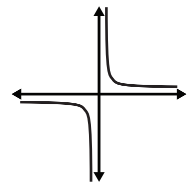
e) Complete the table of values and draw the graph for this experiment.

V (L)	***P*** (kPa)
0.5	
1.0	
2.0	
5.0	
10.0	

Pressure V.S. Volume of 0.011 mol of a gas at 273.15 K

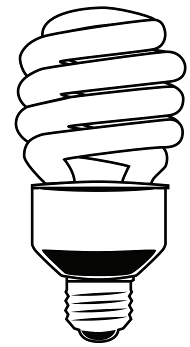


f) Do the results from the table match the predictions in parts b & c?



$y = \frac{1}{x}$

Example 10 *Physics Application: Light Illuminance*



Objects close to a light source appear brighter than objects farther away. This phenomenon is due to the *illuminance* of light, a measure of how much light is incident on a surface. The illuminance of light can be described with the reciprocal-square relation:

$$\boxed{I(d) = \frac{S}{4\pi d^2}}$$

where I is the illuminance (SI unit = lux), S is the amount of light emitted by a source (SI unit = lumens), and d is the distance from the light source in metres.

In an experiment to investigate the reciprocal-square nature of light illuminance, a screen can be moved from a baseline position to various distances from the bulb.

a) If the original distance of the screen from the bulb is doubled, how does the illuminance change?

b) If the original distance of the screen from the bulb is tripled, how does the illuminance change?

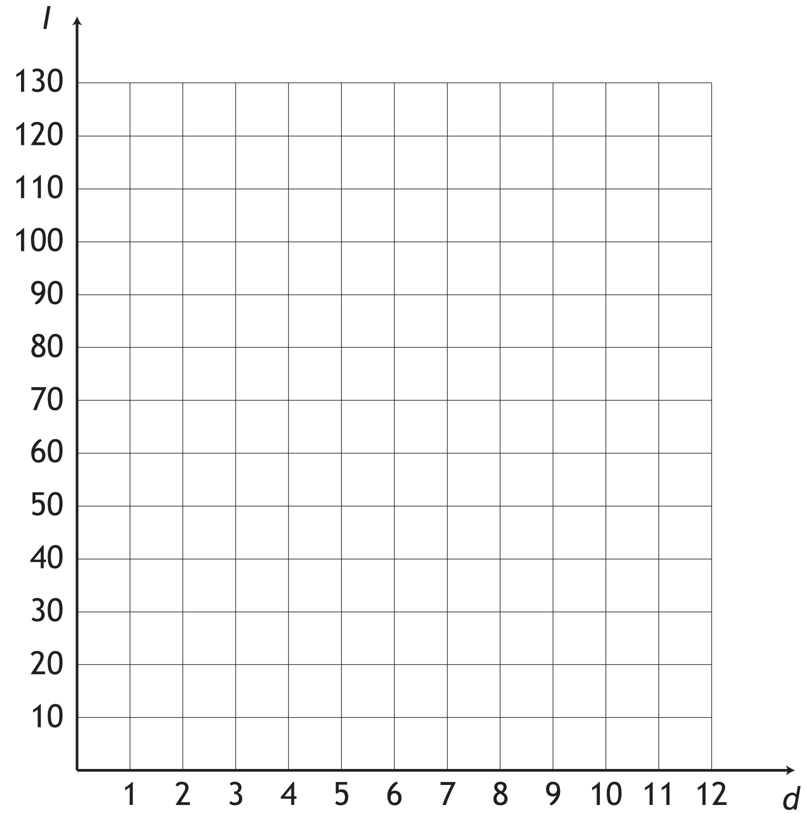
c) If the original distance of the screen from the bulb is halved, how does the illuminance change?

d) If the original distance of the screen from the bulb is quartered, how does the illuminance change?

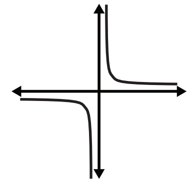
e) A typical household fluorescent bulb emits 1600 lumens. If the original distance from the bulb to the screen was 4 m, complete the table of values and draw the graph.

d (m)	***I*** (W/m^2)
1	
2	
4 ORIGINAL	
8	
12	

Illuminance V.S. Distance for a Fluorescent Bulb



f) Do the results from the table match the predictions made in parts a-d?



$y = \frac{1}{x}$

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Lesson Notes

Example 1

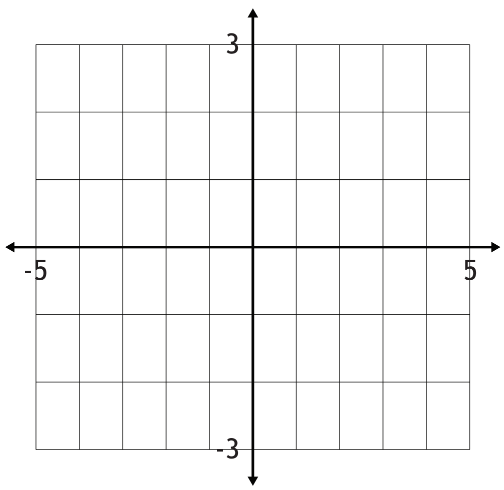
Numerator Degree < Denominator Degree

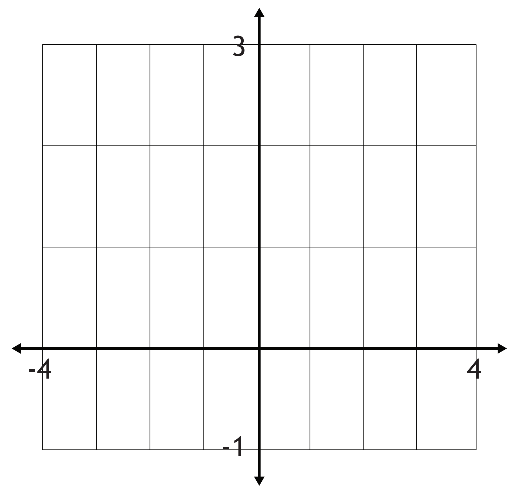
$\frac{\text{Numerator}}{\text{Denominator}}$

Predict if any asymptotes or holes are present in the graph of each rational function. Use a graphing calculator to draw the graph and verify your prediction.

a) $y = \frac{x}{x^2 - 9}$

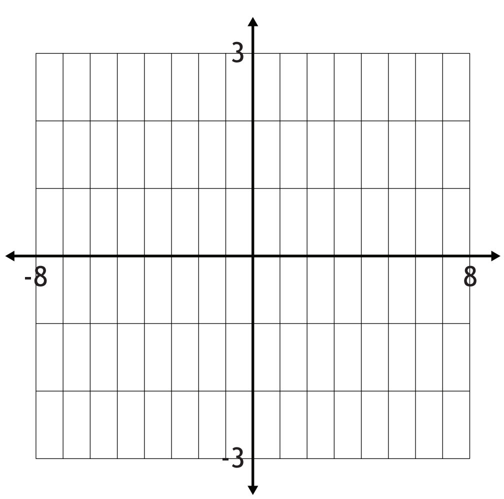
b) $y = \frac{x + 2}{x^2 + 1}$

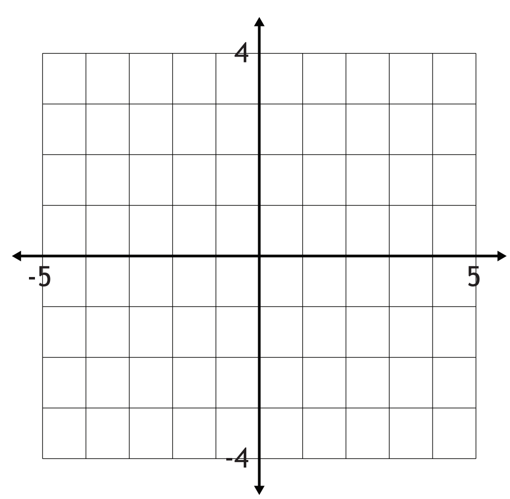


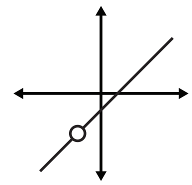


c) $y = \frac{x + 4}{x^2 - 16}$

d) $y = \frac{x^2 - x - 2}{x^3 - x^2 - 2x}$







$$y = \frac{x^2 + x - 2}{x + 2}$$

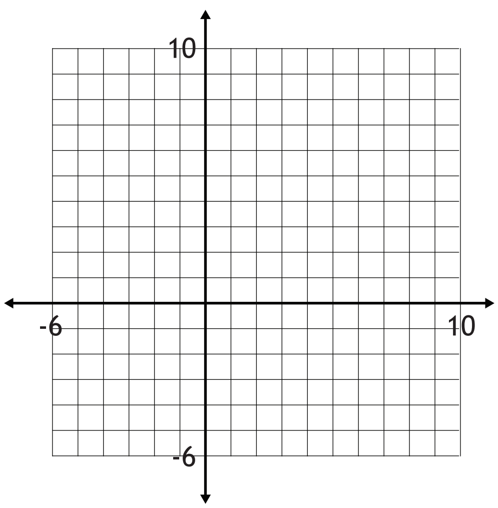
Example 2

Numerator Degree = Denominator Degree

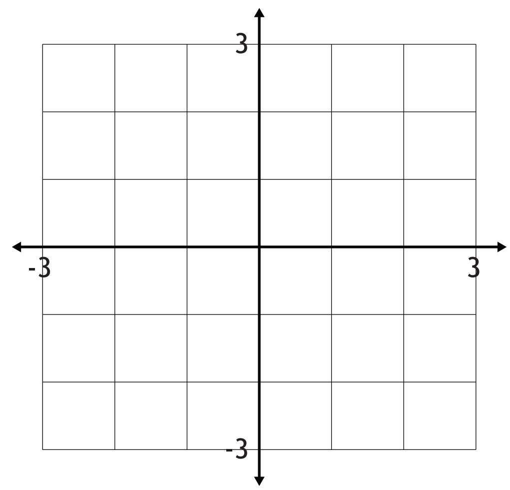
$$\frac{\text{Numerator}}{\text{Denominator}}$$

Predict if any asymptotes or holes are present in the graph of each rational function. Use a graphing calculator to draw the graph and verify your prediction.

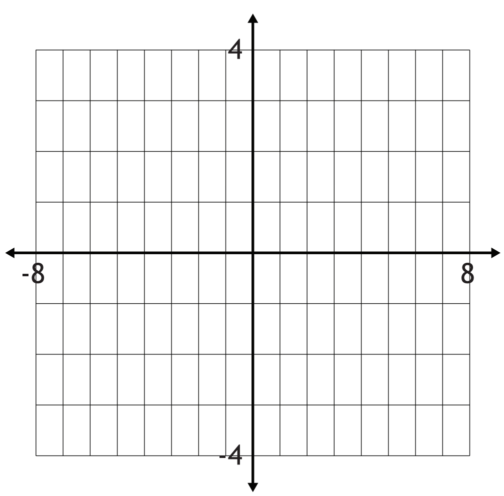
a) $y = \dfrac{4x}{x - 2}$



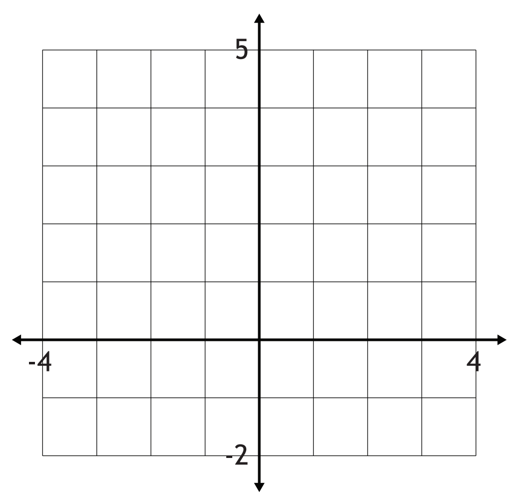
b) $y = \dfrac{x^2}{x^2 - 1}$

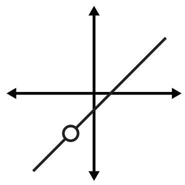


c) $y = \dfrac{3x^2}{x^2 + 9}$



d) $y = \dfrac{3x^2 - 3x - 18}{x^2 - x - 6}$





$y = \dfrac{x^2 + x - 2}{x + 2}$

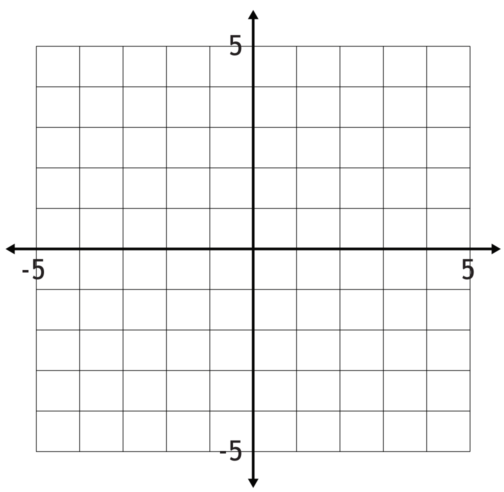
Example 3

Numerator Degree > Denominator Degree

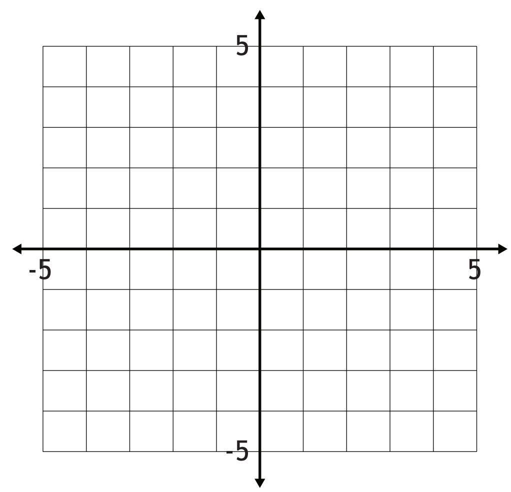
$\dfrac{\text{Numerator}}{\text{Denominator}}$

Predict if any asymptotes or holes are present in the graph of each rational function. Use a graphing calculator to draw the graph and verify your prediction.

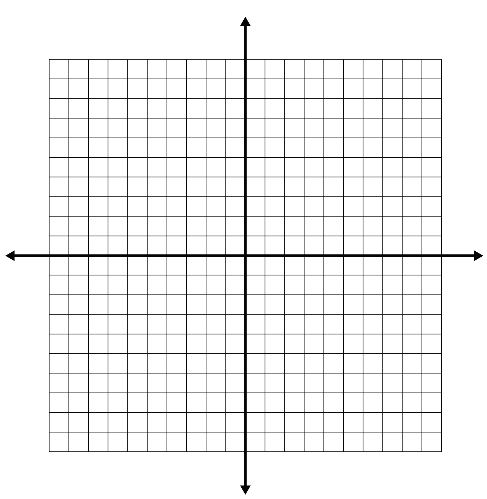
a) $y = \dfrac{x^2 + 5x + 4}{x + 4}$



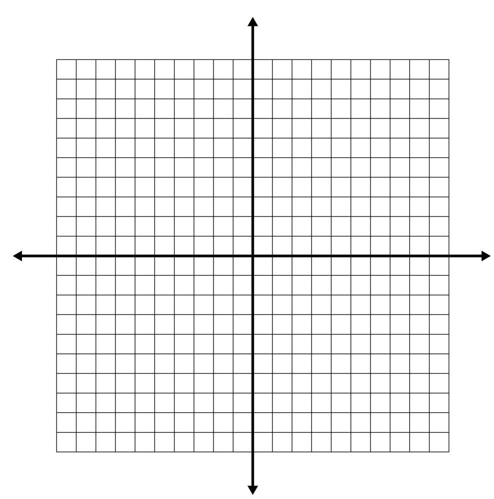
b) $y = \dfrac{x^2 - 4x + 3}{x - 3}$



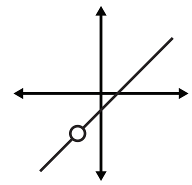
c) $y = \dfrac{x^2 + 5}{x - 1}$



d) $y = \dfrac{x^2 - x - 6}{x + 1}$



Lesson Notes



$$y = \frac{x^2 + x - 2}{x + 2}$$

Example 4

Graph $y = \frac{x}{x^2 - 16}$ without using the graphing feature of your calculator.

Properties of Rational Function Graphs

i) Horizontal Asymptote:

ii) Vertical Asymptote(s):

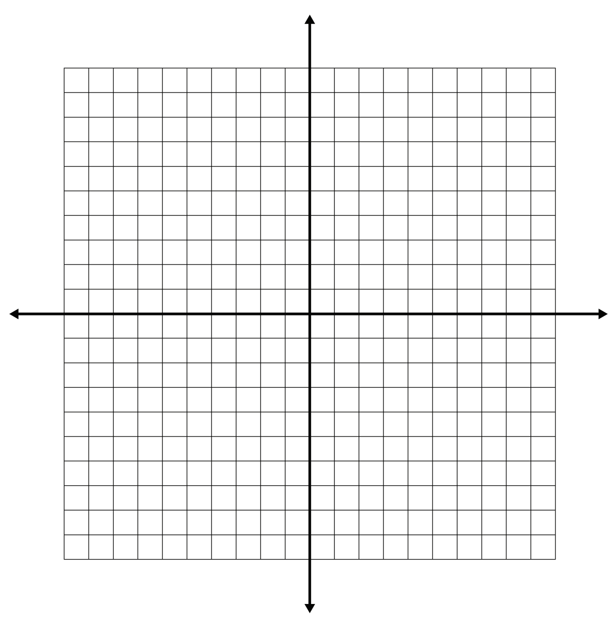
iii) y - intercept:

iv) x - intercept(s):

v) Domain and Range:

Other Points:

These are any extra points required to shape the graph. You may use your calculator to evaluate these.



Example 5 Graph $y = \dfrac{2x - 6}{x + 2}$ without using the graphing feature of your calculator.

Properties of Rational Function Graphs

i) Horizontal Asymptote:

ii) Vertical Asymptote(s):

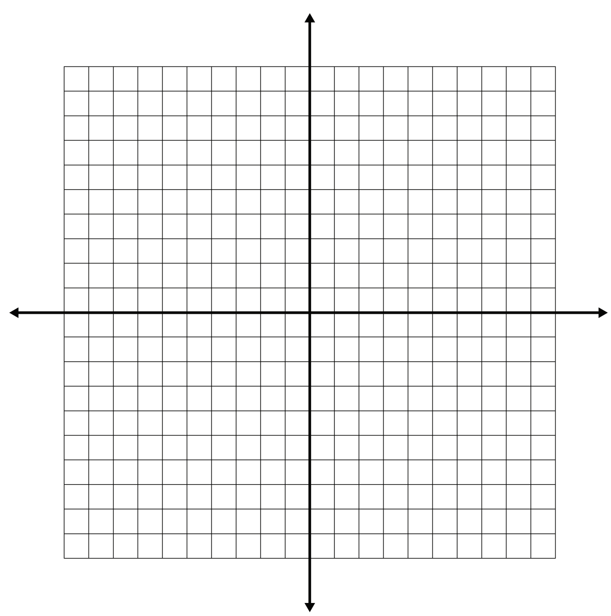
iii) y - intercept:

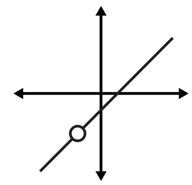
iv) x - intercept(s):

v) Domain and Range:

Other Points:

These are any extra points required to shape the graph.
You may use your calculator to evaluate these.





$$y = \frac{x^2 + x - 2}{x + 2}$$

Example 6 Graph $y = \frac{x^2 + 2x - 8}{x - 1}$ without using the graphing feature of your calculator.

Properties of Rational Function Graphs

i) Horizontal Asymptote:

Other Points:

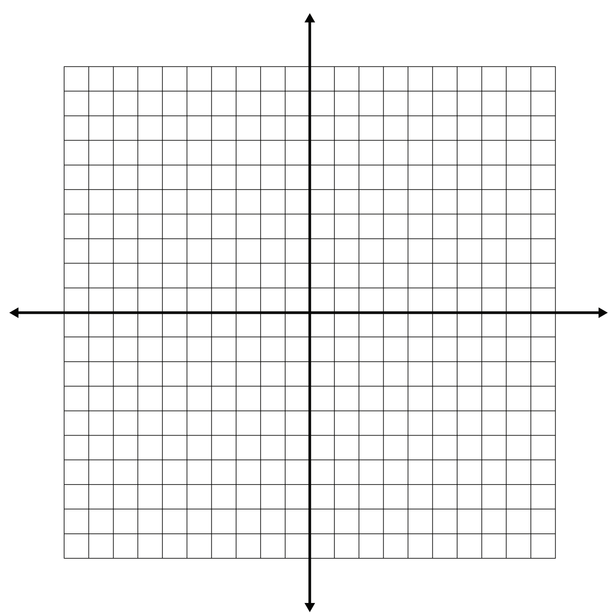
These are any extra points required to shape the graph. You may use your calculator to evaluate these.

ii) Vertical Asymptote(s):

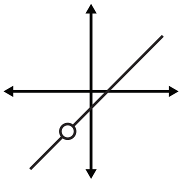
iii) y - intercept:

iv) x - intercept(s):

v) Domain and Range:



vi) Oblique Asymptote



$y = \dfrac{x^2 + x - 2}{x + 2}$

Polynomial, Radical, and Rational Functions
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Example 7

Graph $y = \dfrac{x^2 - 5x + 6}{x - 2}$ without using the graphing feature of your calculator.

Properties of Rational Function Graphs

i) Can this rational function be simplified?

ii) Holes:

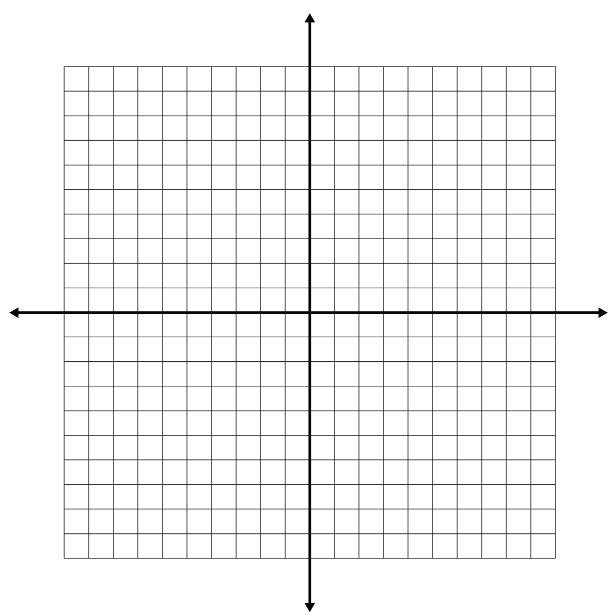
iii) y - intercept:

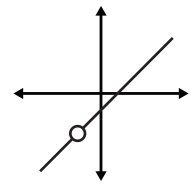
iv) x - intercept(s):

v) Domain and Range:

Other Points:

These are any extra points required to shape the graph. You may use your calculator to evaluate these.





$$y = \frac{x^2 + x - 2}{x + 2}$$

Example 8

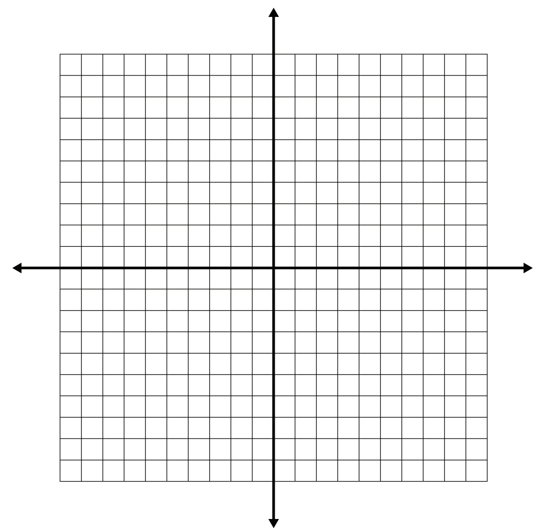
Find the rational function with each set of characteristics and draw the graph.

Finding a Rational Function from its Properties or Graph.

a)

vertical asymptote(s)	$x = -2$, $x = 4$
horizontal asymptote	$y = 1$
x-intercept(s)	$(-3, 0)$ and $(5, 0)$
hole(s)	none

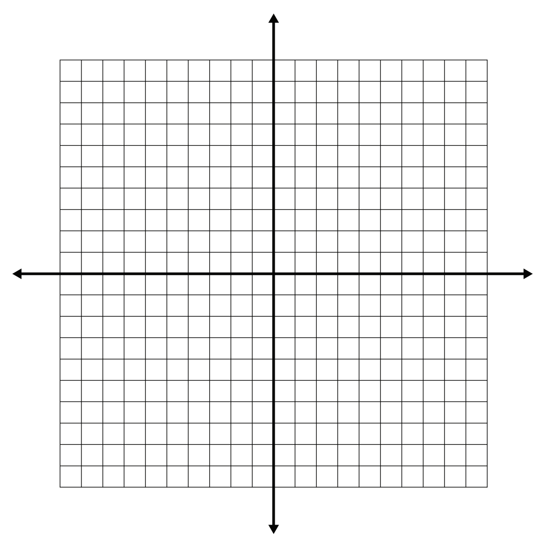
Rational Function:



b)

vertical asymptote(s)	$x = 0$
horizontal asymptote	$y = 0$
x-intercept(s)	none
hole(s)	$(-1, -1)$

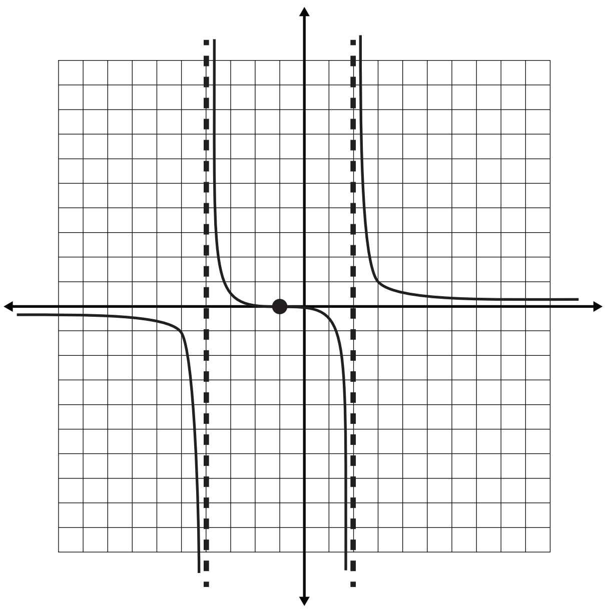
Rational Function:



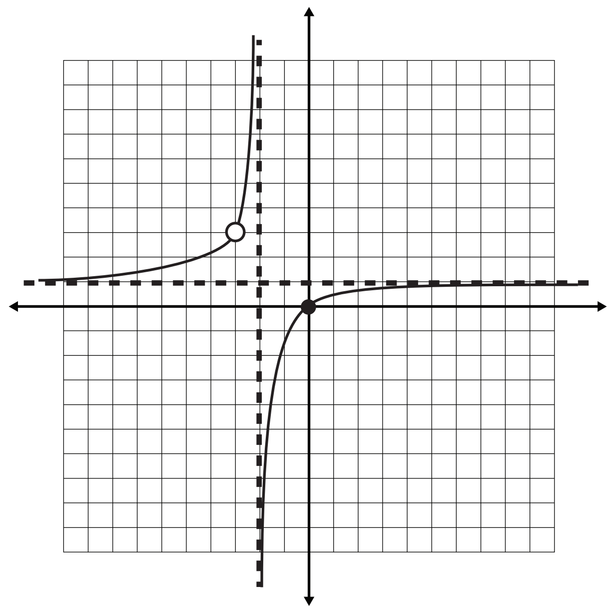
Example 9 Find the rational function shown in each graph.

Finding a Rational Function from its Properties or Graph.

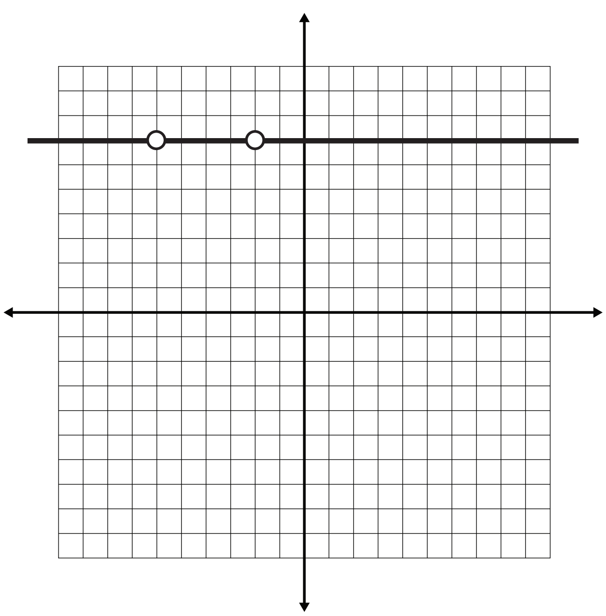
a)



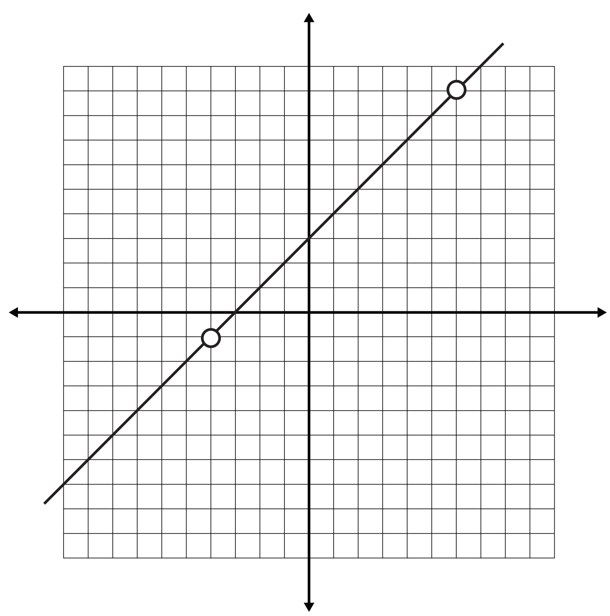
b)

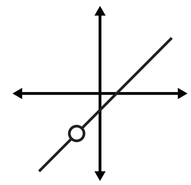


c)



d)





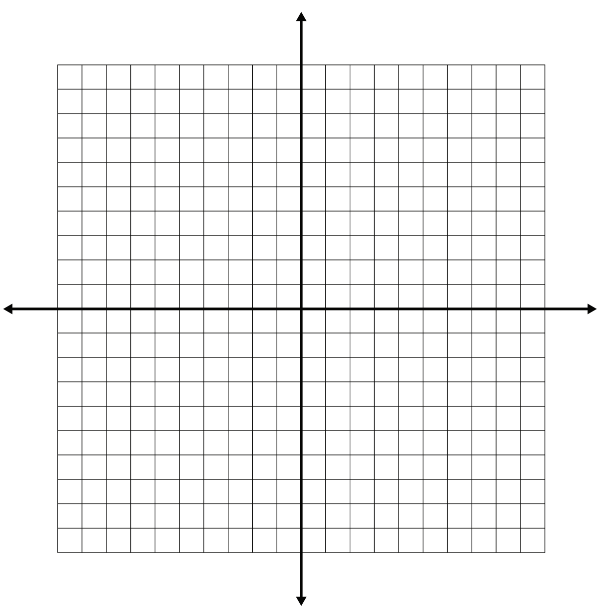
$$y = \frac{x^2 + x - 2}{x + 2}$$

Example 10 Solve the rational equation $\frac{3x}{x - 1} = 4$ in three different ways.

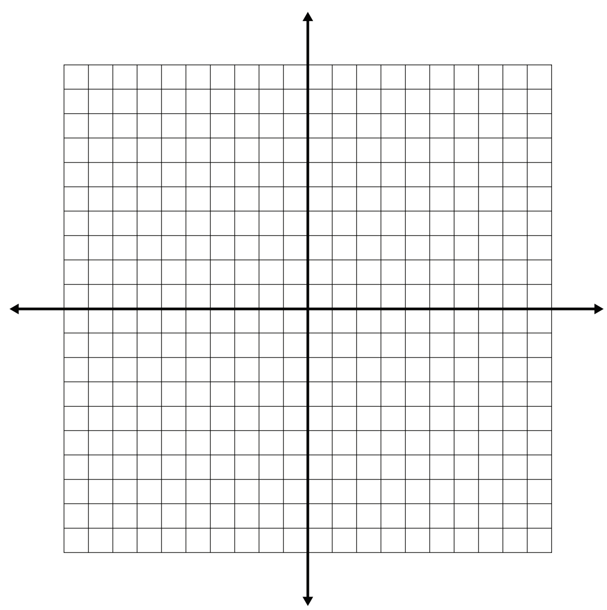
Rational Equations

a) Solve algebraically and check for extraneous roots.

b) Solve the equation by finding the point of intersection of a system of functions.



c) Solve the equation by finding the x-intercept(s) of a single function.

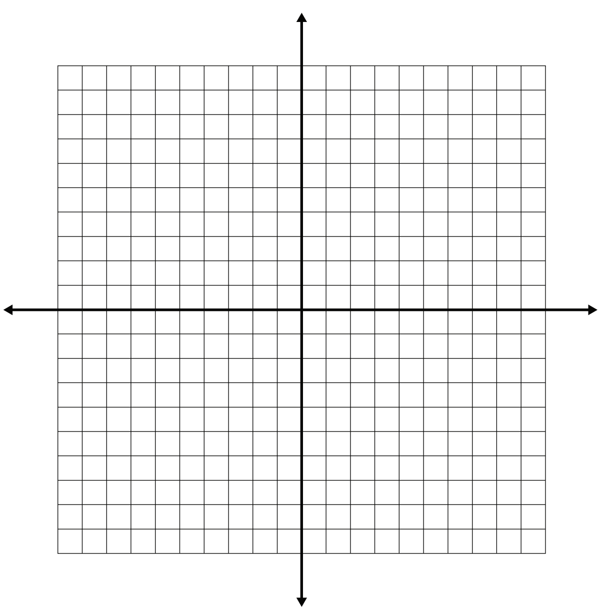


Example 11 Solve the rational equation $\frac{6}{x} - \frac{9}{x - 1} = -6$ in three different ways.

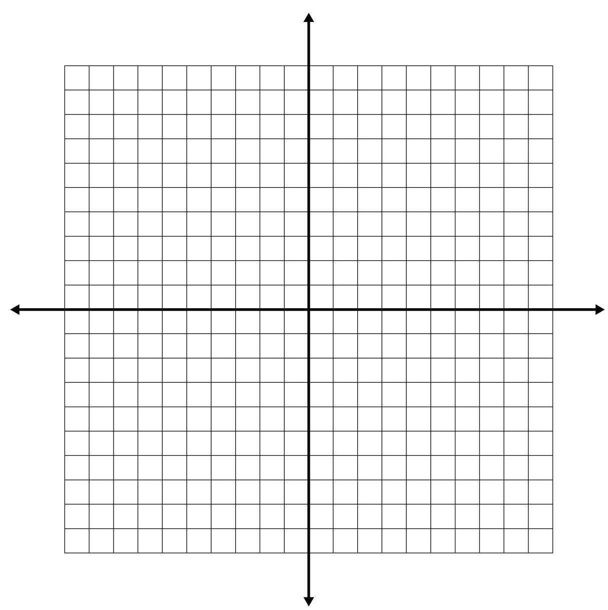
Rational Equations

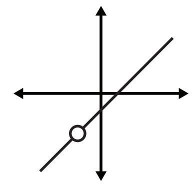
a) Solve algebraically and check for extraneous roots.

b) Solve the equation by finding the point of intersection of a system of functions.



c) Solve the equation by finding the x-intercept(s) of a single function.





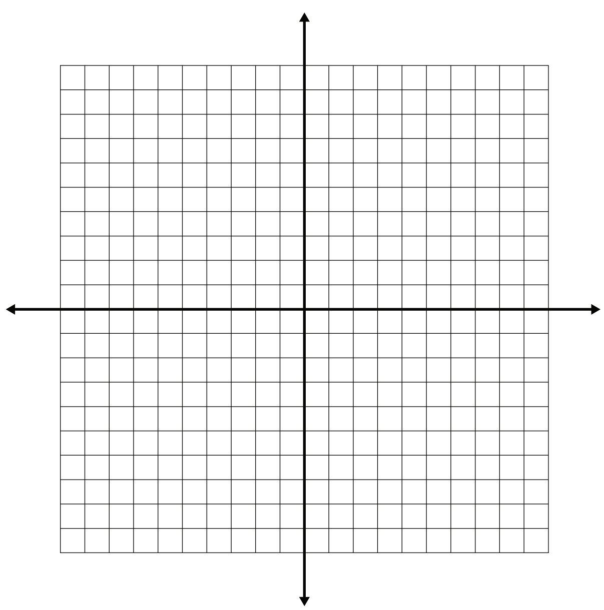
$y = \dfrac{x^2 + x - 2}{x + 2}$

Example 12 Solve the equation $\dfrac{x}{x-2} - \dfrac{4}{x+1} = \dfrac{6}{x^2 - x - 2}$ in three different ways.

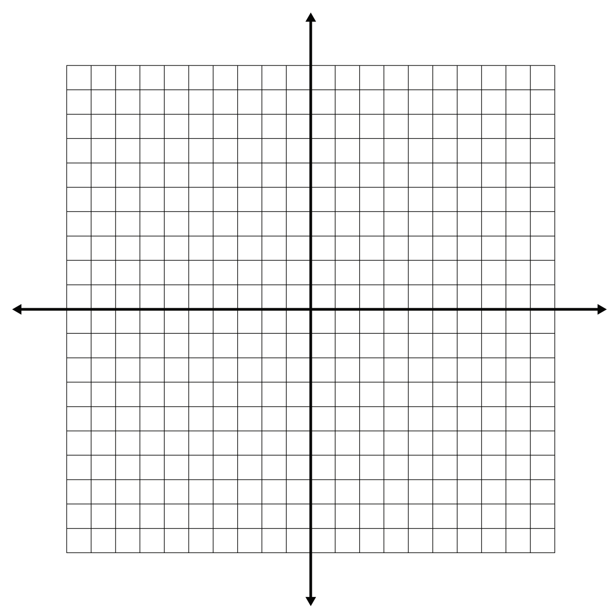
Rational Equations

a) Solve algebraically and check for extraneous roots.

b) Solve the equation by finding the point of intersection of a system of functions.



c) Solve the equation by finding the x-intercept(s) of a single function.



Example 13

Cynthia jogs 3 km/h faster than Alan. In a race, Cynthia was able to jog 15 km in the same time it took Alan to jog 10 km. How fast were Cynthia and Alan jogging?

a) Fill in the table and derive an equation that can be used to solve this problem.

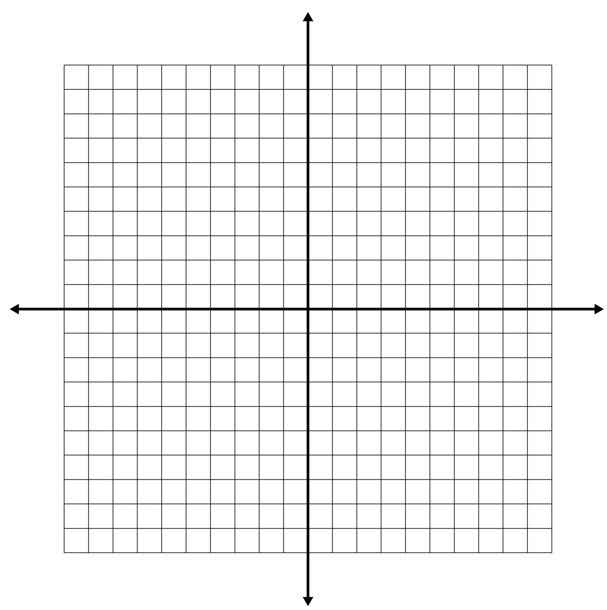
	d	*s*	*t*
Cynthia			
Alan			

b) Solve algebraically.

c) Check your answer by either:
i) finding the point of intersection of two functions.

OR

ii) finding the x-intercept(s) of a single function.



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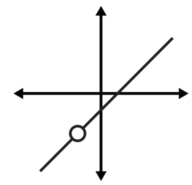


$$y = \frac{x^2 + x - 2}{x + 2}$$

Example 14

George can canoe 24 km downstream and return to his starting position (upstream) in 5 h. The speed of the current is 2 km/h. What is the speed of the canoe in still water?

a) Fill in the table and derive an equation that can be used to solve this problem.

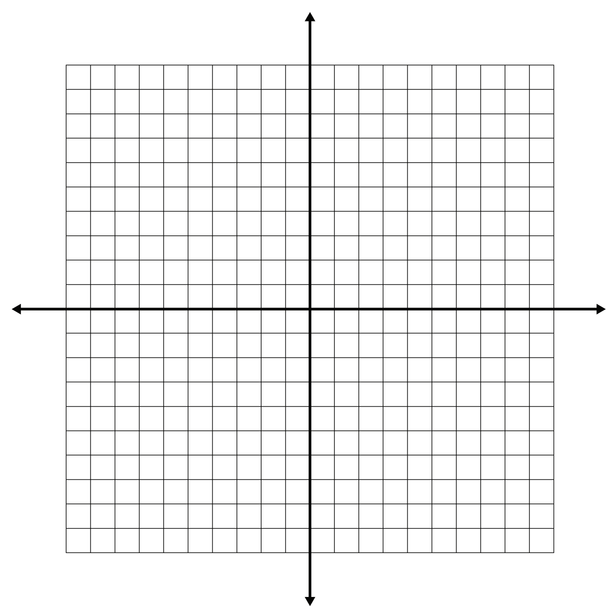
	d	*s*	*t*
Upstream			
Downstream			

b) Solve algebraically.

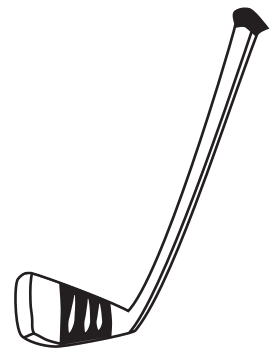
c) Check your answer by either:
i) finding the point of intersection of two functions.

OR

ii) finding the x-intercept(s) of a single function.



Example 15



The shooting percentage of a hockey player is ratio of scored goals to total shots on goal. So far this season, Laura has scored 2 goals out of 14 shots taken. Assuming Laura scores a goal with every shot from now on, how many goals will she need to have a 40% shooting percentage?

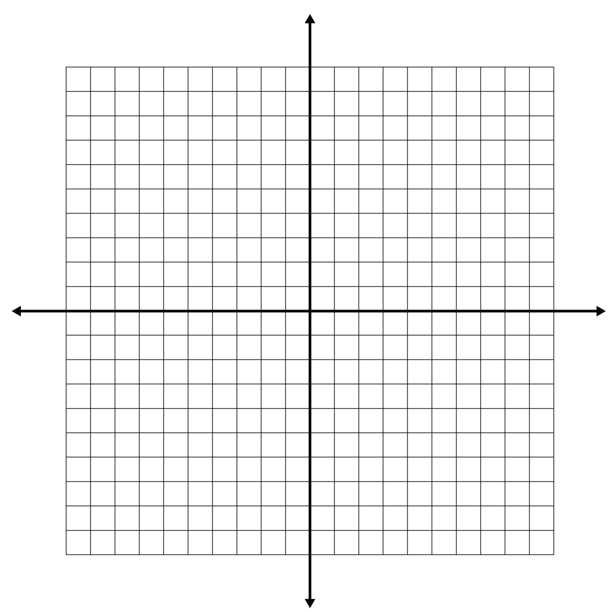
a) Derive an equation that can be used to solve this problem.

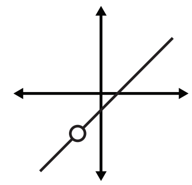
b) Solve algebraically.

c) Check your answer by either:
i) finding the point of intersection of two functions.

OR

ii) finding the x-intercept(s) of a single function.





$$y = \frac{x^2 + x - 2}{x + 2}$$

Example 16

A 300 g mixture of nuts contains peanuts and almonds.
The mixture contains 35% almonds by mass.
What mass of almonds must be added to this mixture so it contains 50% almonds?

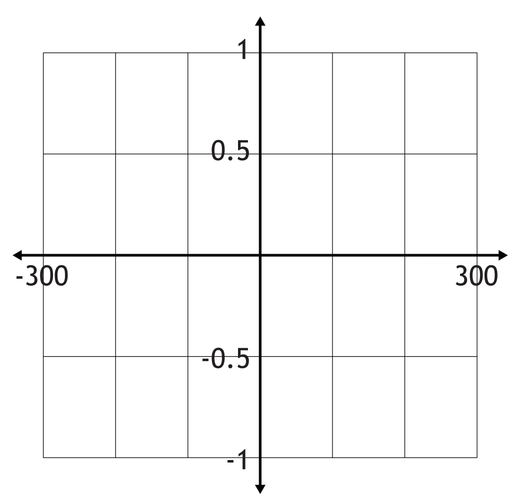
a) Derive an equation that can be used to solve this problem.

b) Solve algebraically.

c) Check your answer by either:
i) finding the point of intersection of two functions.

OR

ii) finding the x-intercept(s) of a single function.

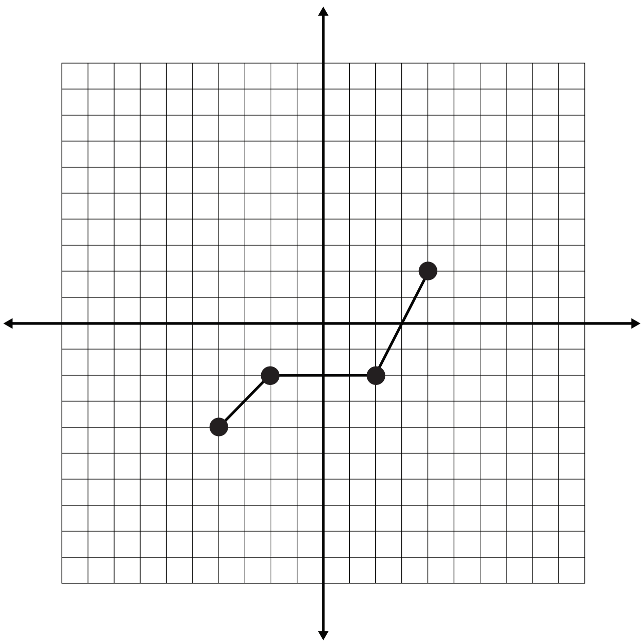


Example 1 Draw the graph resulting from each transformation. Label the invariant points.

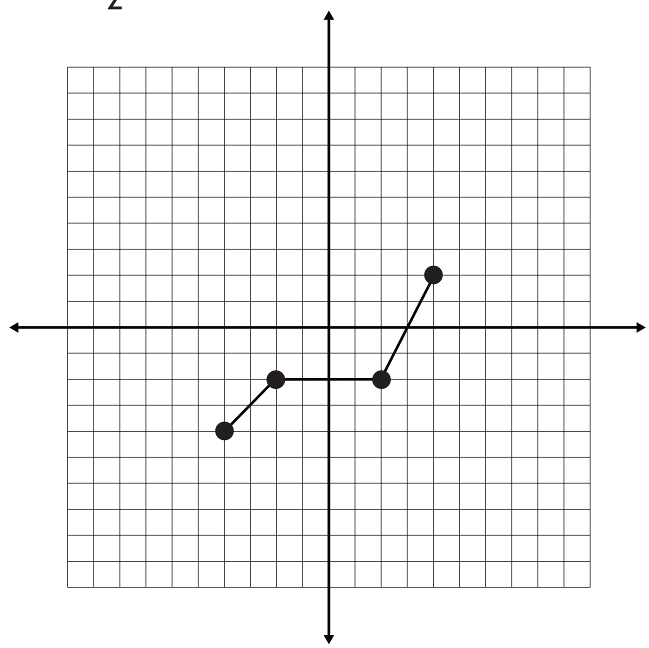
Graphing Stretches

Vertical Stretches

a) $y = 2f(x)$

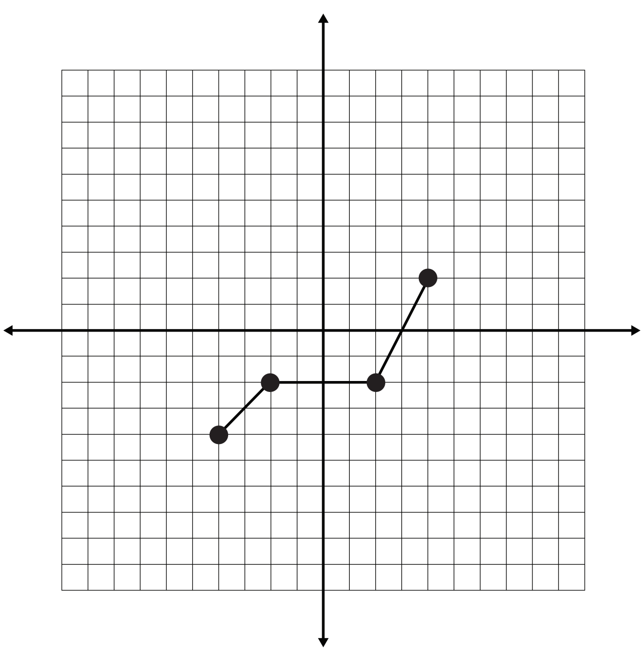


b) $y = \frac{1}{2}f(x)$

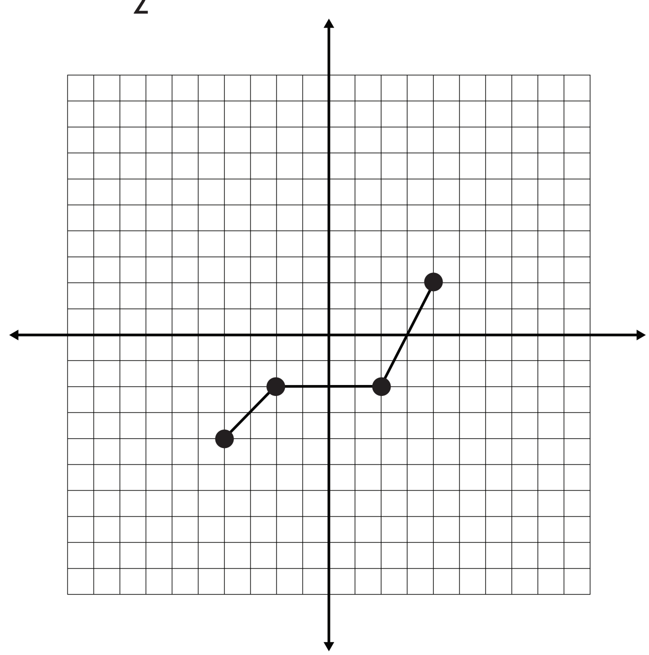


Horizontal Stretches

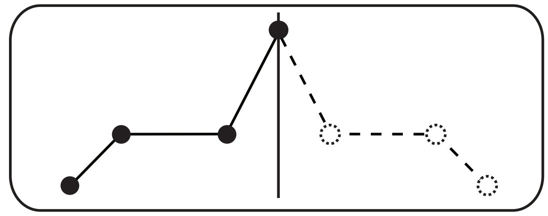
c) $y = f(2x)$



d) $y = f(\frac{1}{2}x)$



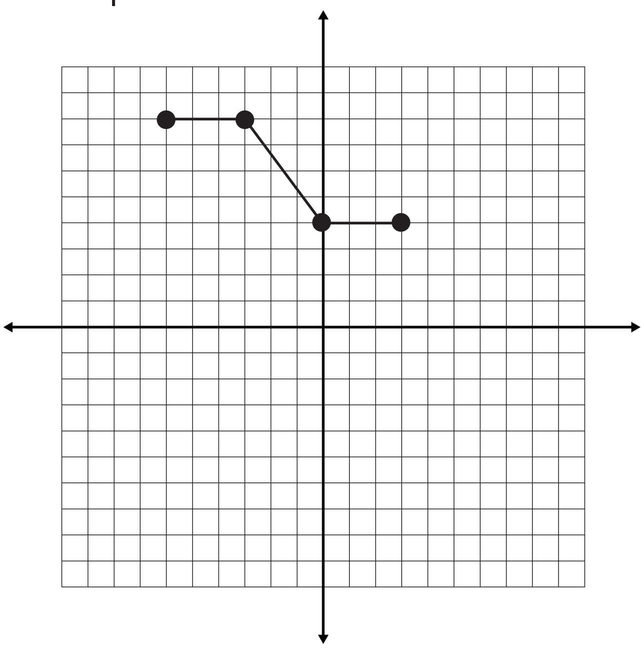
Transformations and Operations
LESSON ONE - *Basic Transformations*
Lesson Notes



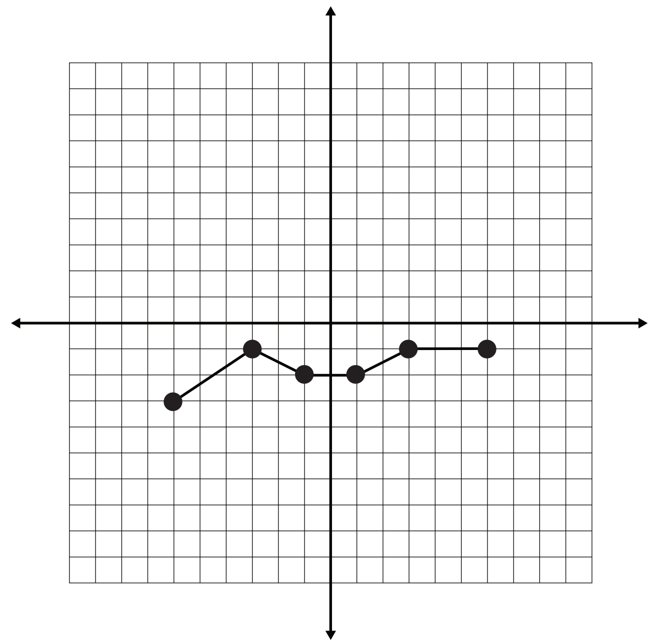
Example 2 Draw the graph resulting from each transformation.
Label the invariant points.

Graphing Stretches

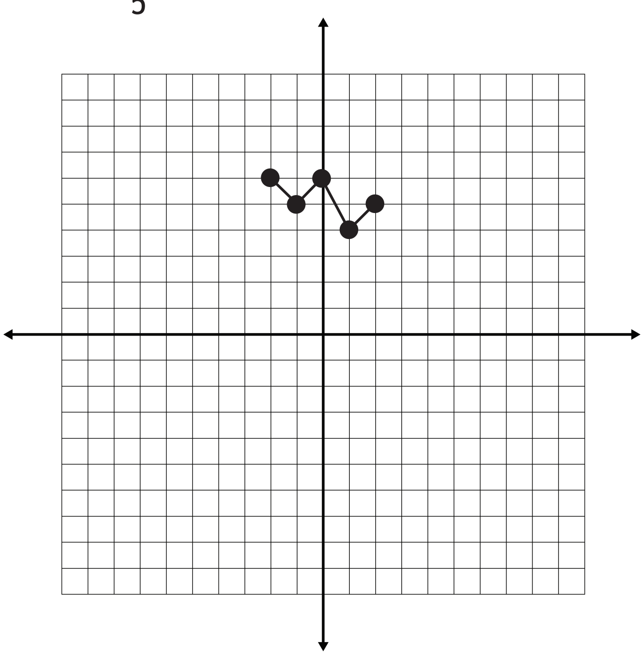
a) $y = \frac{1}{4}f(x)$



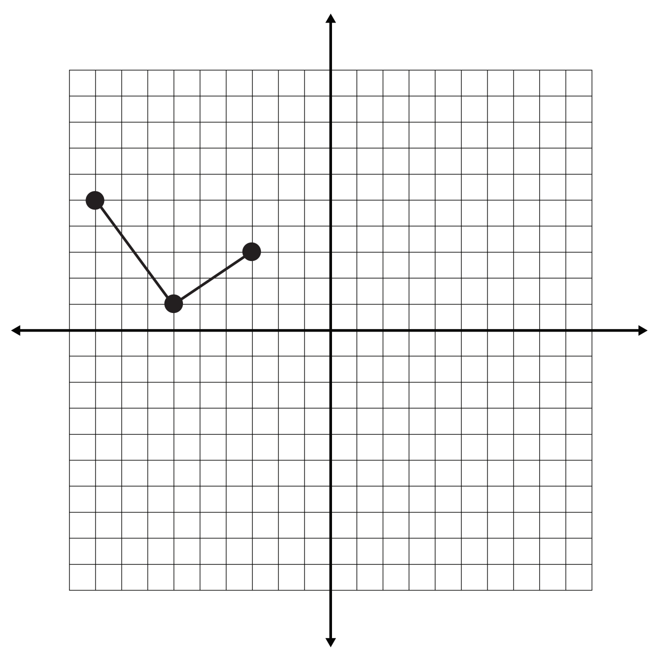
b) $y = 3f(x)$

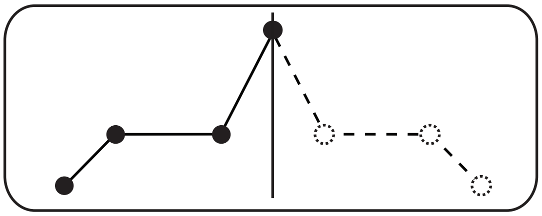


c) $y = f(\frac{1}{5}x)$



d) $y = f(3x)$





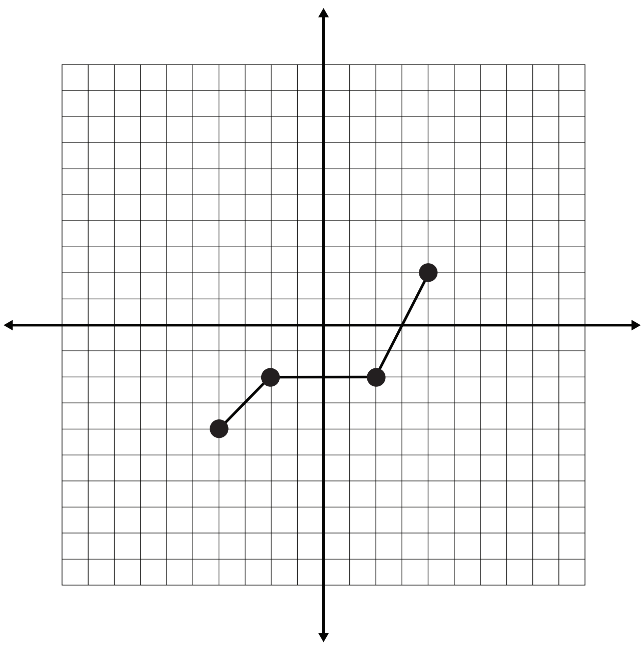
Transformations and Operations
LESSON ONE - *Basic Transformations*
Lesson Notes

Example 3 Draw the graph resulting from each transformation.
Label the invariant points.

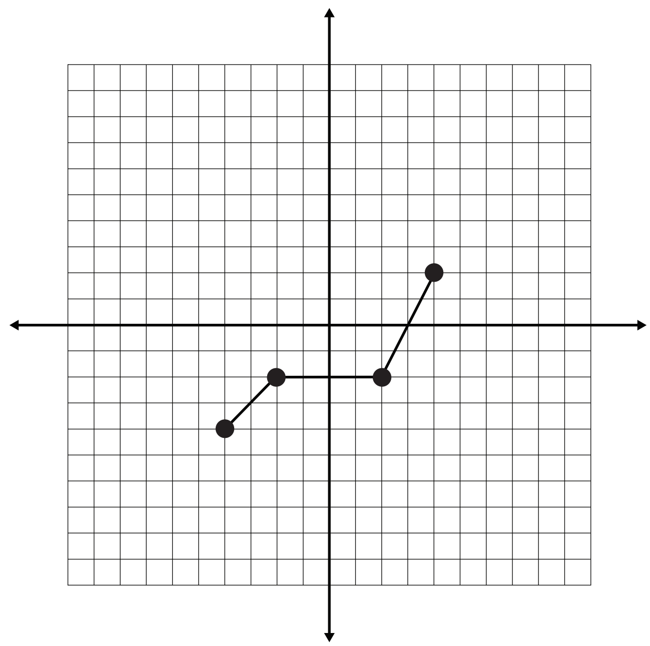
Graphing Reflections

Reflections

a) $y = -f(x)$

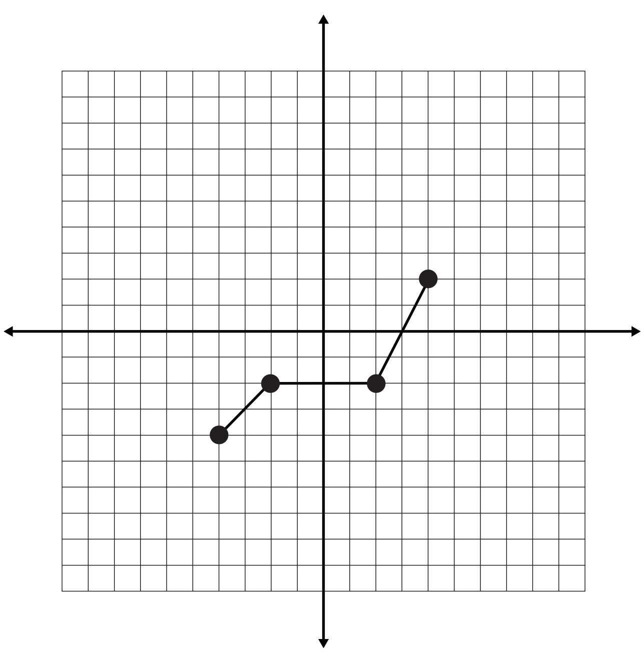


b) $y = f(-x)$

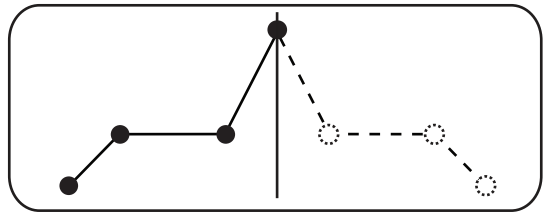


Inverses

c) $x = f(y)$



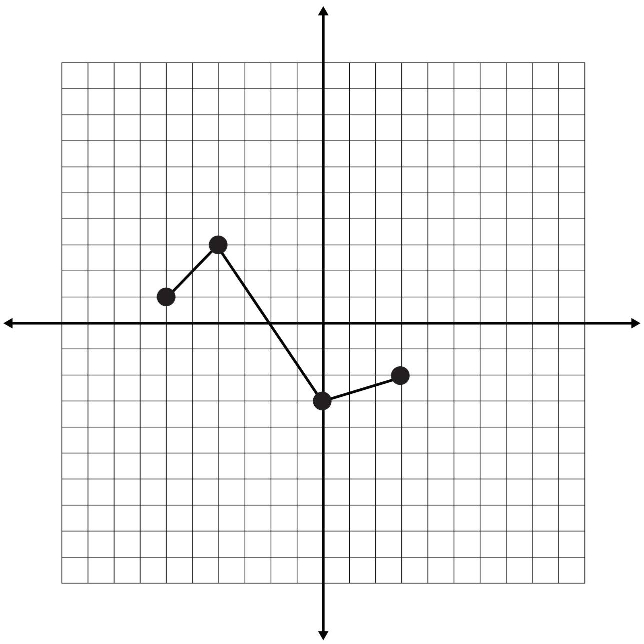
Transformations and Operations
LESSON ONE - *Basic Transformations*
Lesson Notes



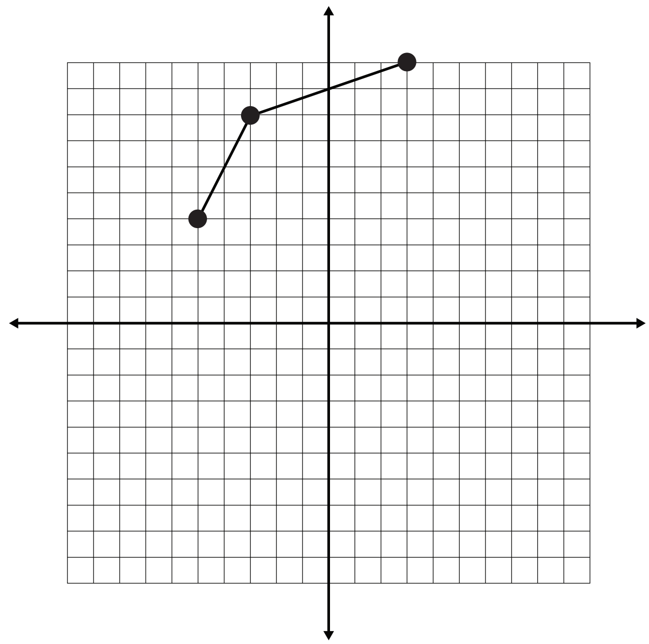
Example 4 Draw the graph resulting from each transformation. Label the invariant points.

Graphing Reflections

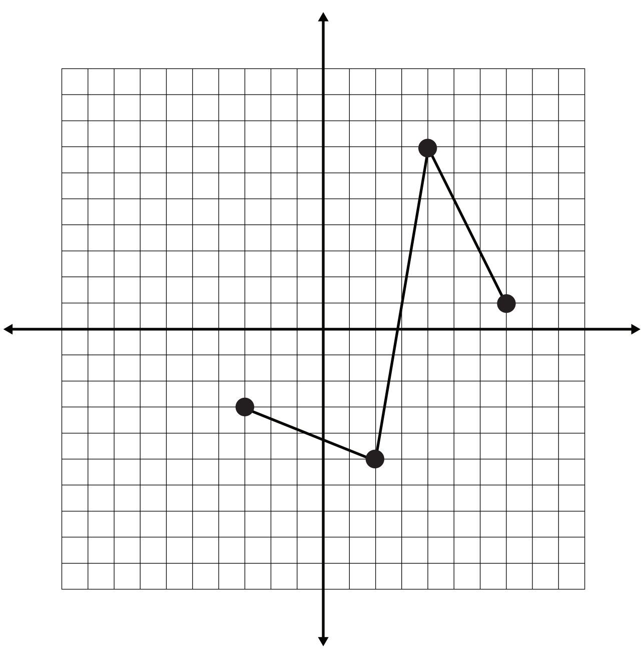
a) y = -f(x)

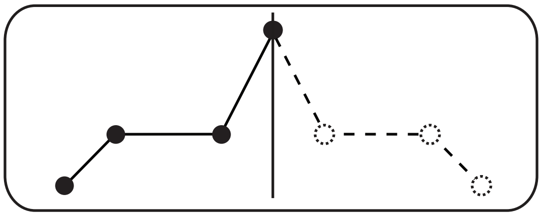


b) y = f(-x)



c) x = f(y)



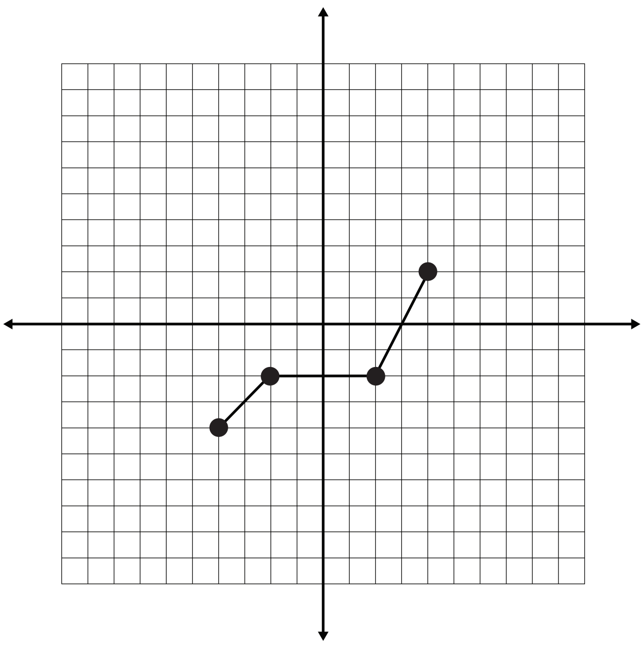


Example 5 Draw the graph resulting from each transformation.

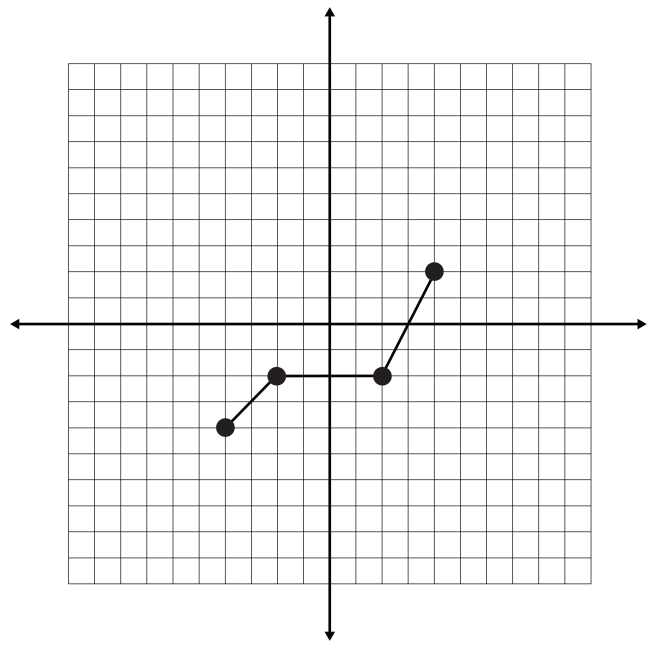
Graphing Translations

Vertical Translations

a) y = f(x) + 3

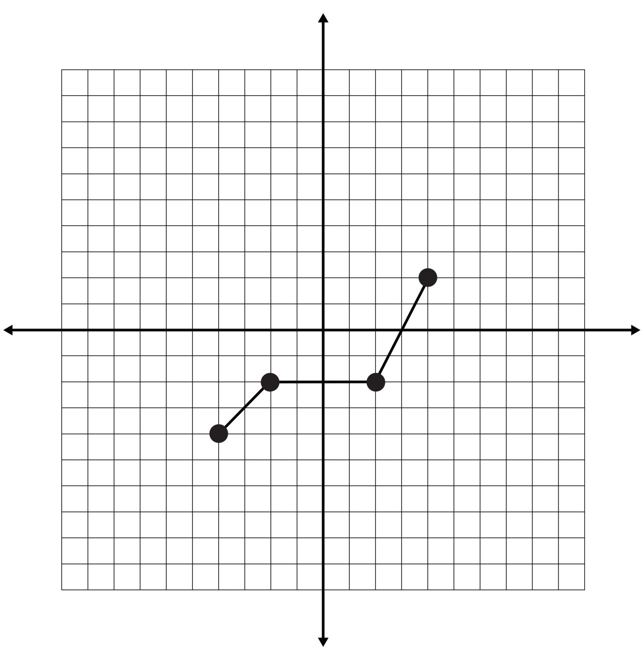


b) y = f(x) - 4

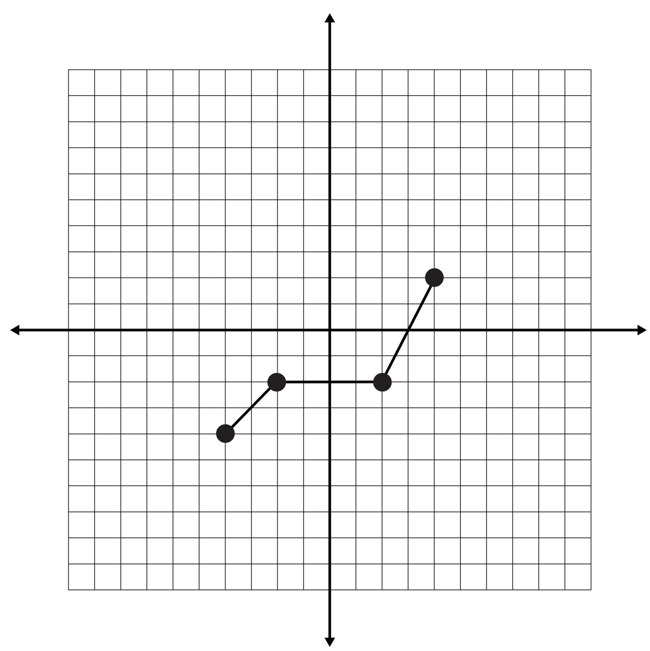


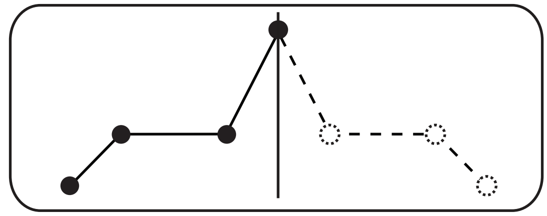
Horizontal Translations

c) y = f(x - 2)



d) y = f(x + 3)

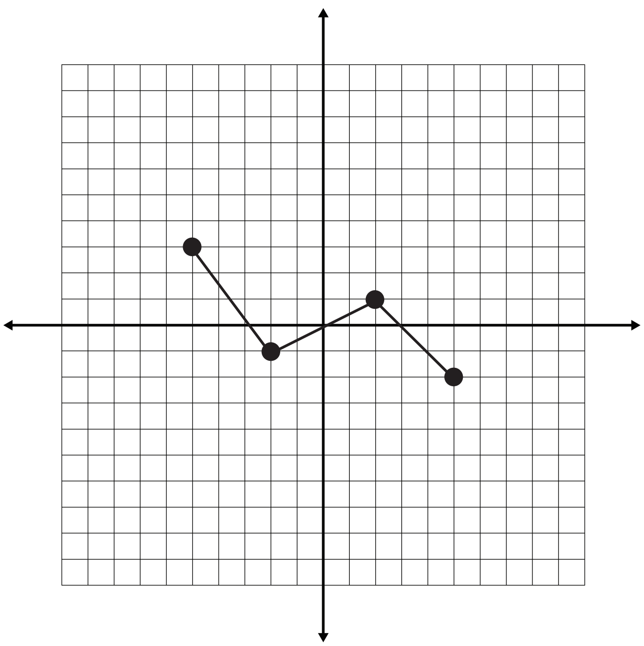




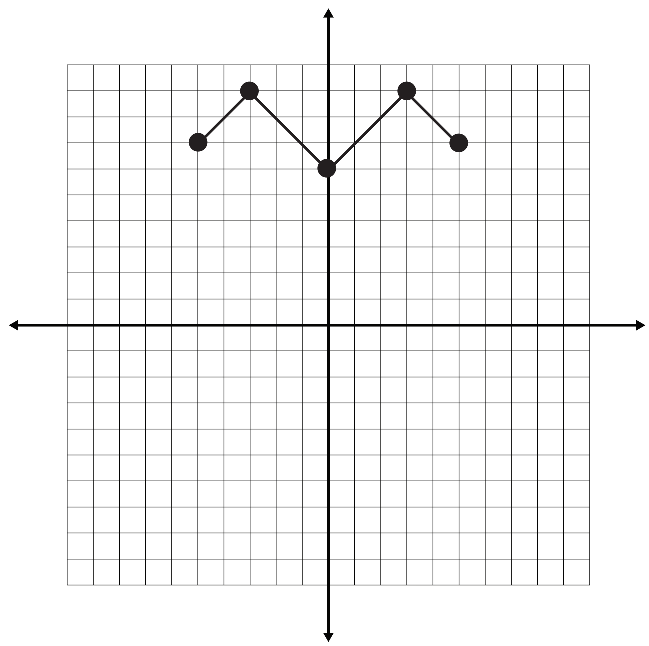
Example 6 Draw the graph resulting from each transformation.

Graphing Translations

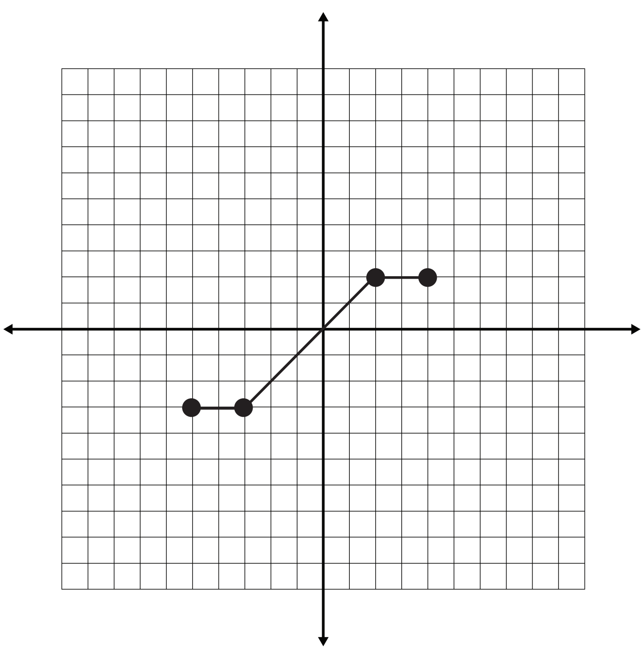
a) $y - 4 = f(x)$



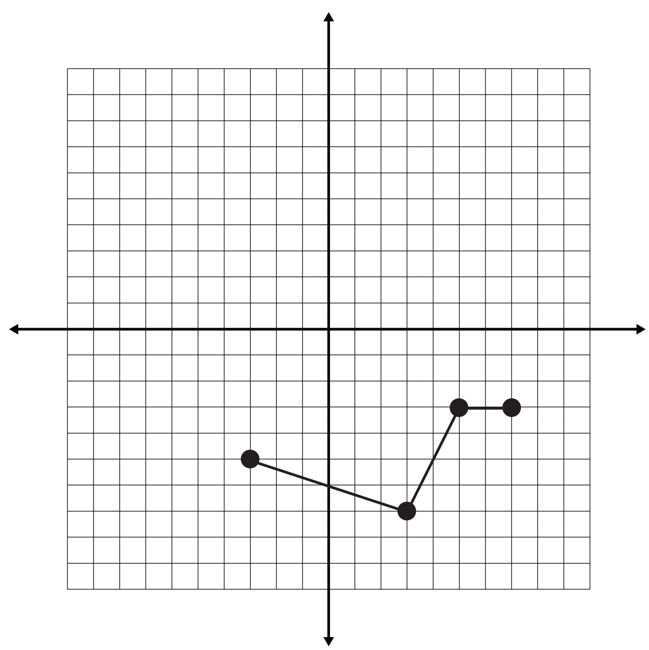
b) $y = f(x) - 3$

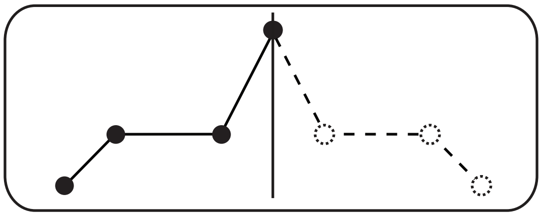


c) $y = f(x - 5)$



d) $y = f(x + 4)$

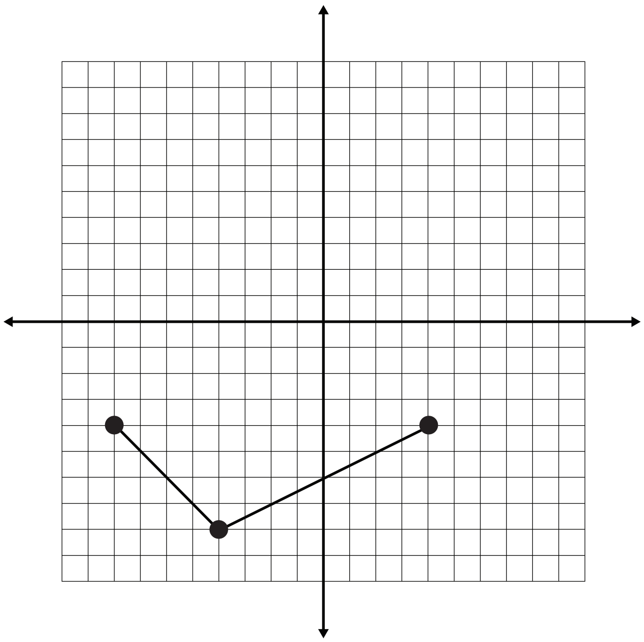




Example 7 Draw the transformed graph. Write the transformation as both an equation and a mapping.

Mappings

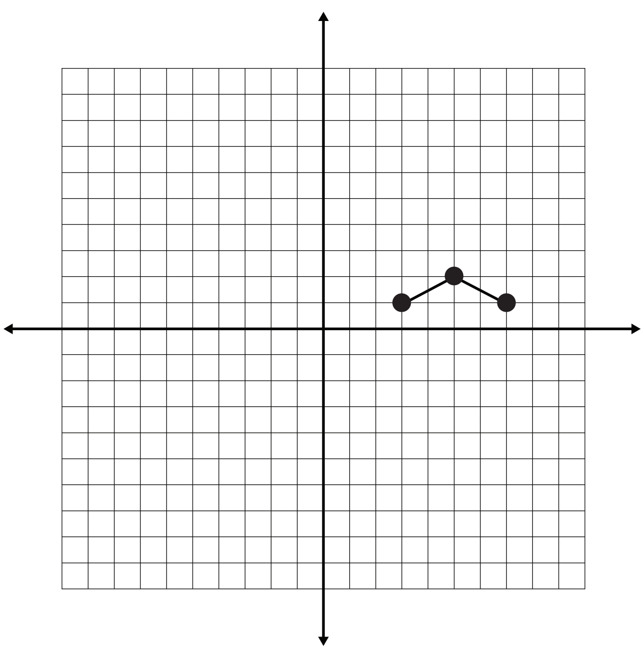
a) The graph of f(x) is horizontally stretched by a factor of $\frac{1}{2}$.



Transformation Equation:

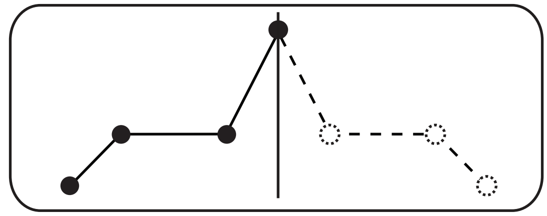
Transformation Mapping:

b) The graph of f(x) is horizontally translated 6 units left.

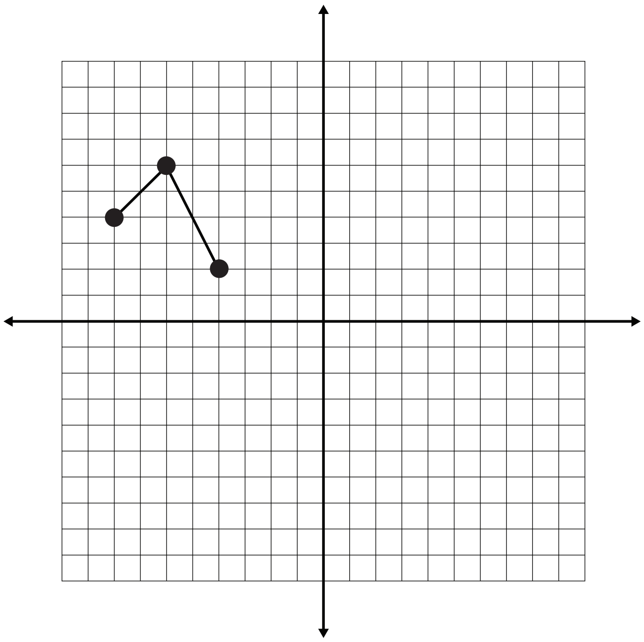


Transformation Equation:

Transformation Mapping:



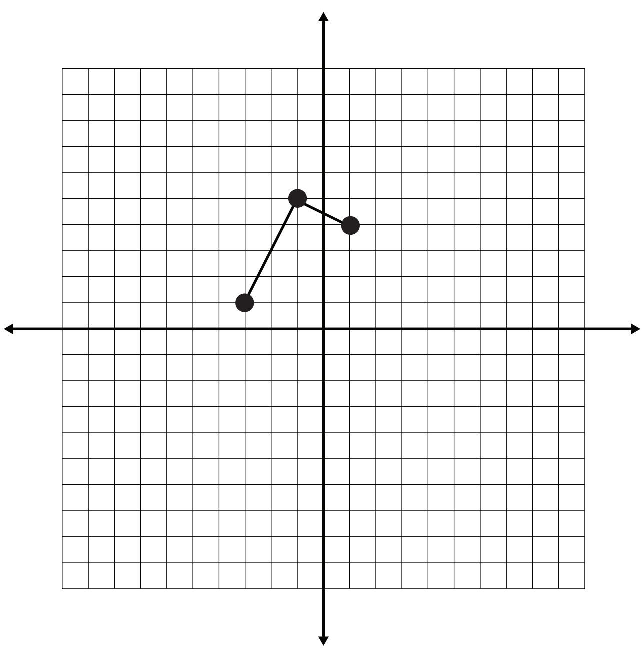
c) The graph of f(x) is vertically translated 4 units down.



Transformation Equation: ____________

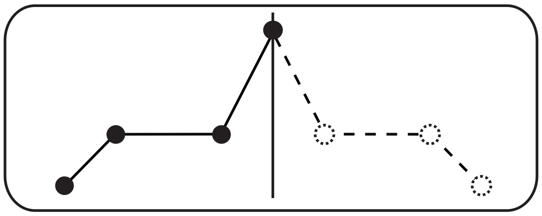
Transformation Mapping: ____________

d) The graph of f(x) is reflected in the x-axis.



Transformation Equation: ____________

Transformation Mapping: ____________



Example 8 Write a sentence describing each transformation, then write the transformation equation.

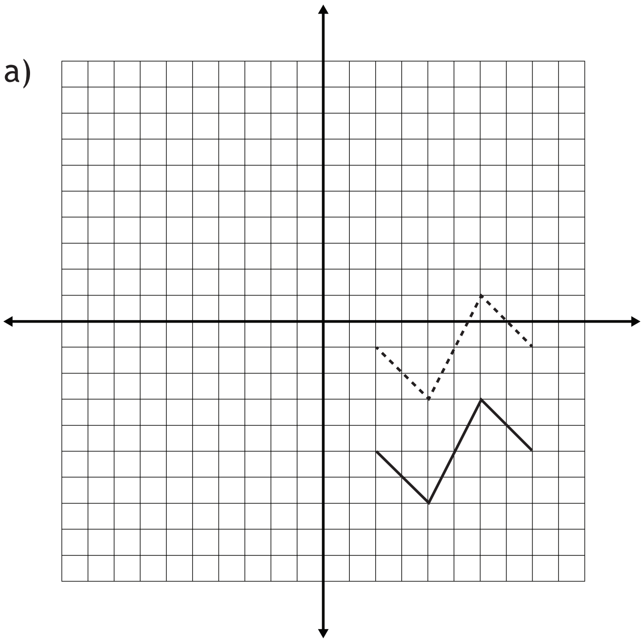
Describing a Transformation

Original graph: --------------

Transformed graph: ————

Think of the dashed line as representing where the graph was in the past, and the solid line is where the graph is now.

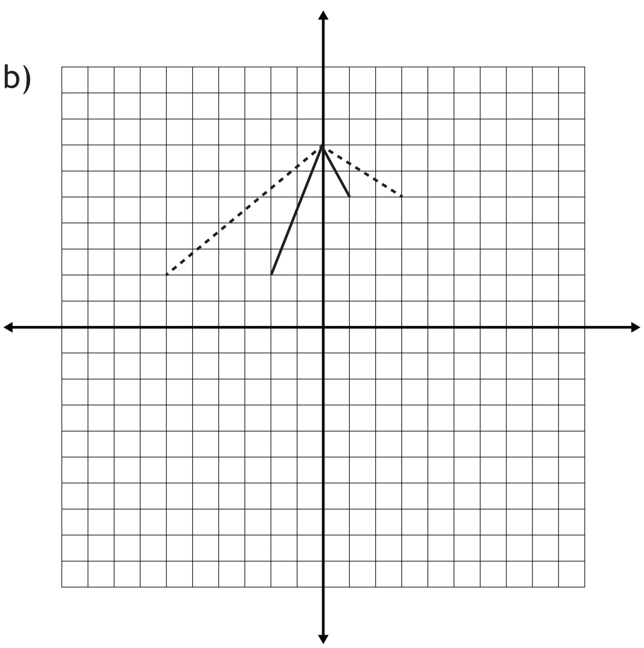
a)



Transformation Equation:

Transformation Mapping:

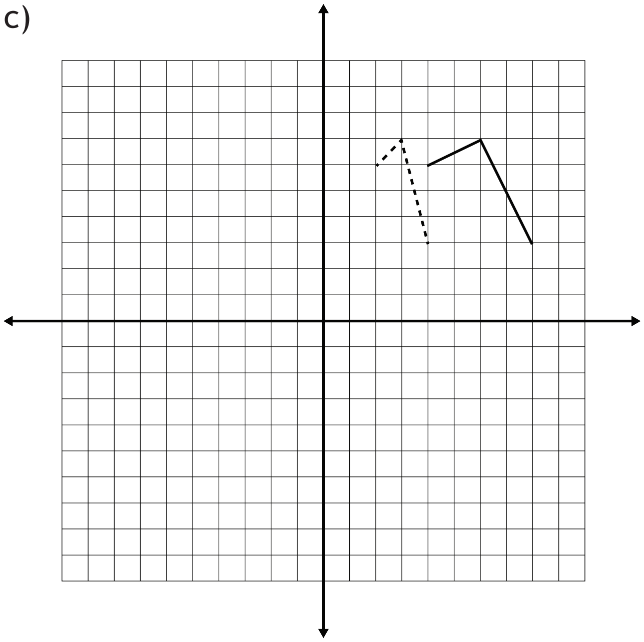
b)



Transformation Equation:

Transformation Mapping:

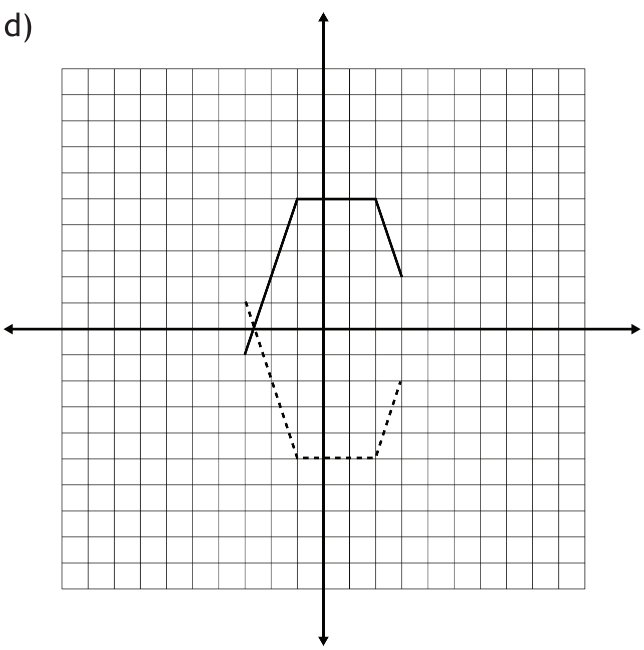
c)



Transformation Equation: ____________

Transformation Mapping: ____________

d)



Transformation Equation: ____________

Transformation Mapping: ____________

Example 9

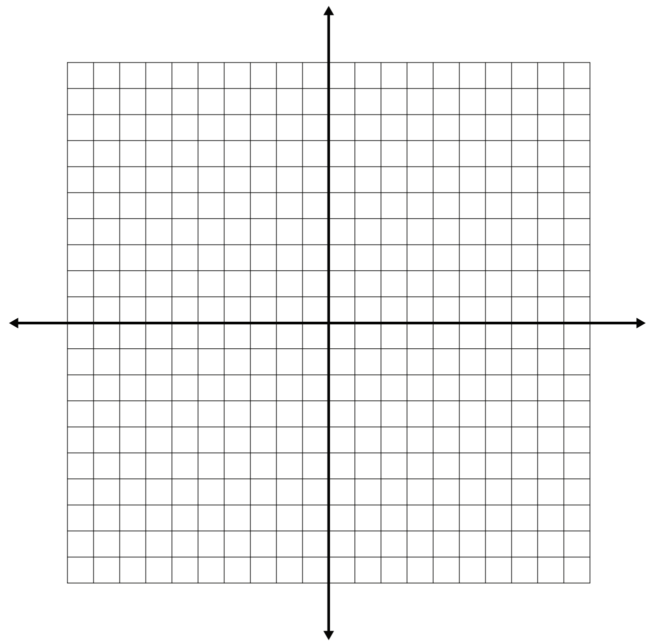
Describe each transformation and derive the equation of the transformed graph. Draw the original and transformed graphs.

Transforming an Existing Function ***(stretches)***

a) Original graph: $f(x) = x^2 - 1$
Transformation: $y = 2f(x)$

Transformation Description:

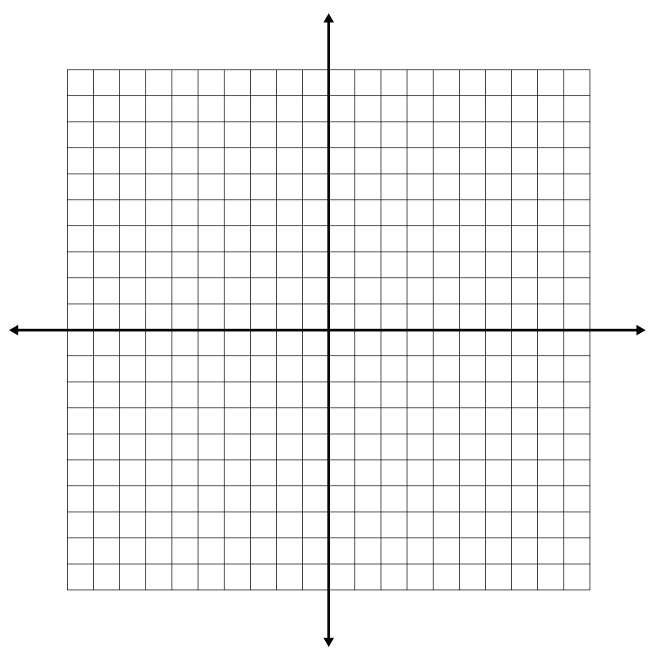
New Function After Transformation:

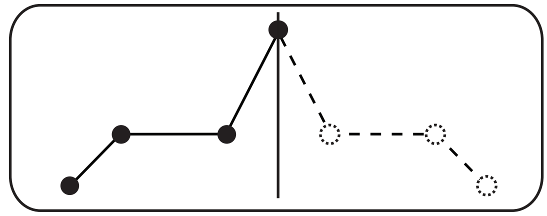


b) Original graph: $f(x) = x^2 + 1$
Transformation: $y = f(2x)$

Transformation Description:

New Function After Transformation:



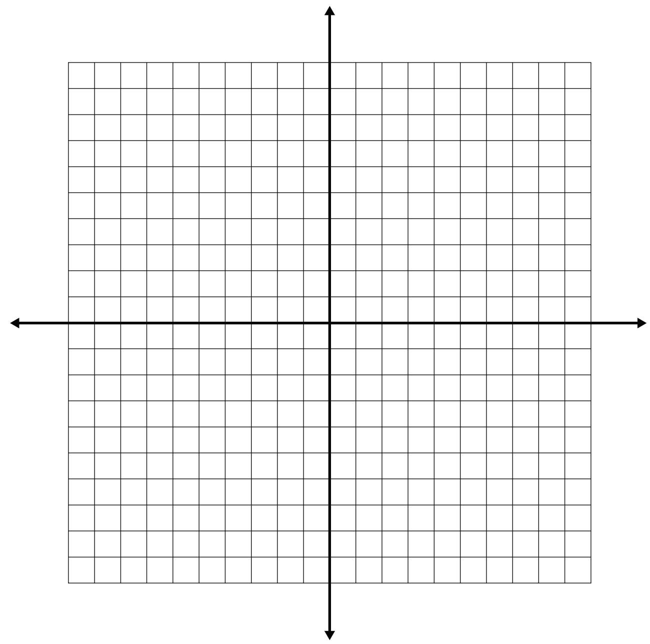


Transforming an Existing Function ***(reflections)***

c) Original graph: $f(x) = x^2 - 2$
Transformation: $y = -f(x)$

Transformation Description:

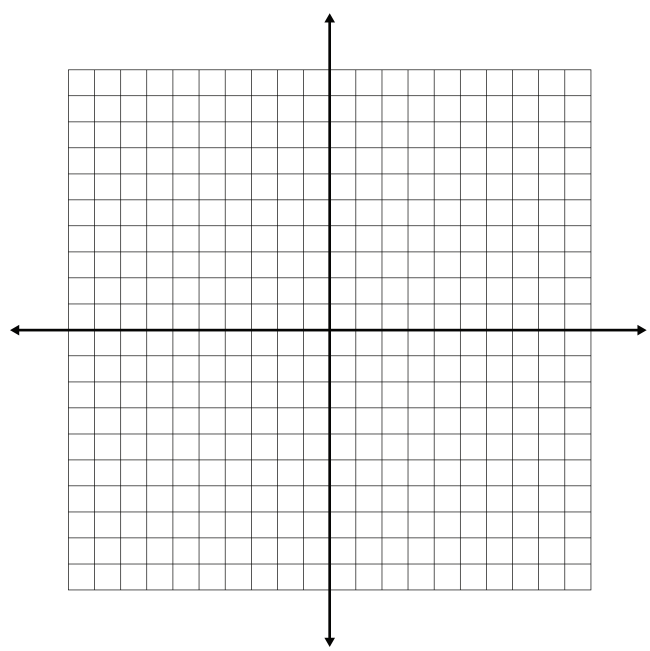
New Function After Transformation:



d) Original graph: $f(x) = (x - 6)^2$
Transformation: $y = f(-x)$

Transformation Description:

New Function After Transformation:



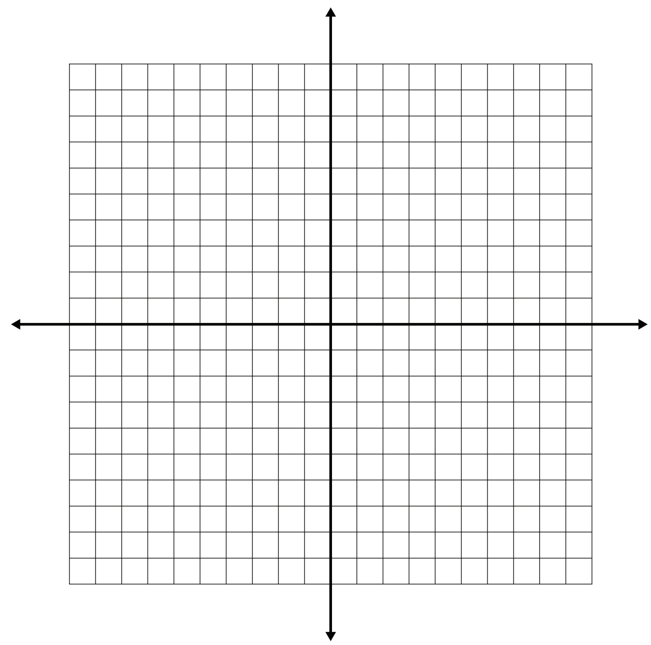
Example 10 Describe each transformation and derive the equation of the transformed graph. Draw the original and transformed graphs.

Transforming an Existing Function
(translations)

a)
Original graph: $f(x) = x^2$
Transformation: $y - 2 = f(x)$

Transformation Description:

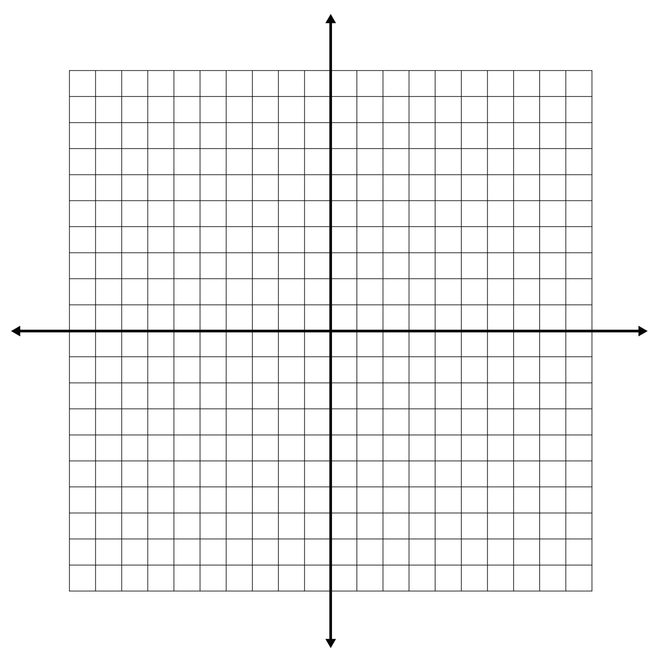
New Function After Transformation:



b)
Original graph: $f(x) = x^2 - 4$
Transformation: $y = f(x) - 4$

Transformation Description:

New Function After Transformation:

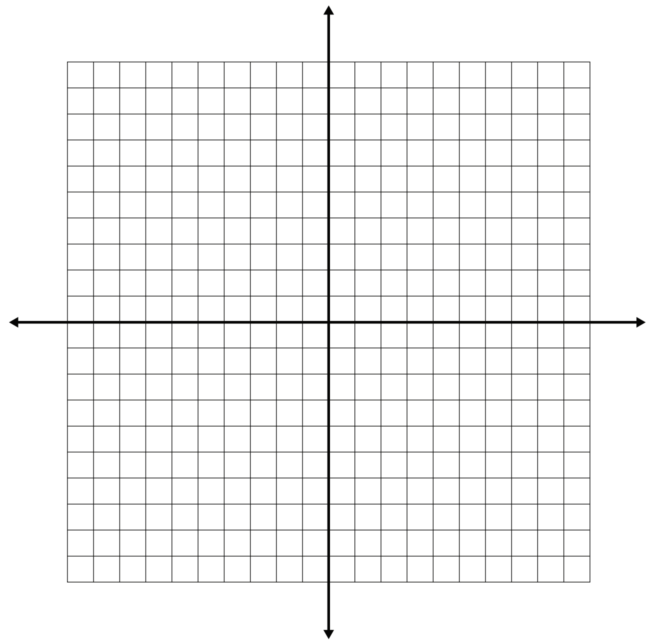


Transforming an Existing Function ***(translations)***

c) Original graph: $f(x) = x^2$
Transformation: $y = f(x - 2)$

Transformation Description:

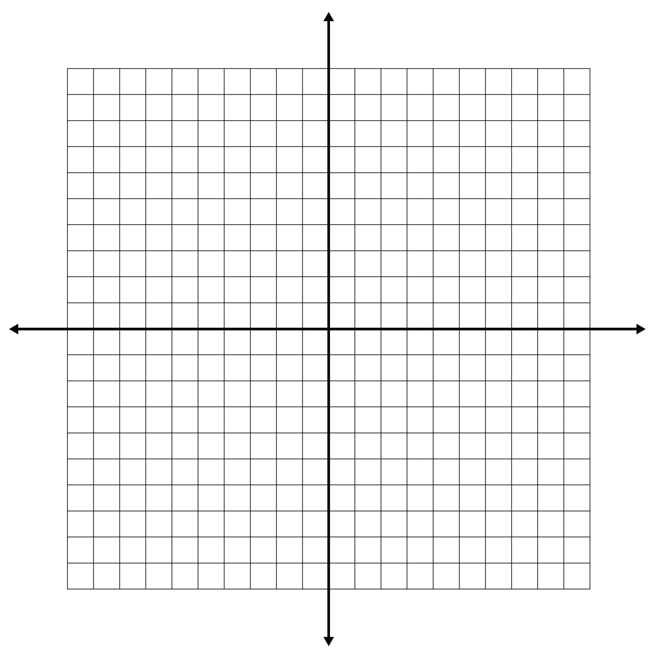
New Function After Transformation:



d) Original graph: $f(x) = (x + 3)^2$
Transformation: $y = f(x - 7)$

Transformation Description:

New Function After Transformation:

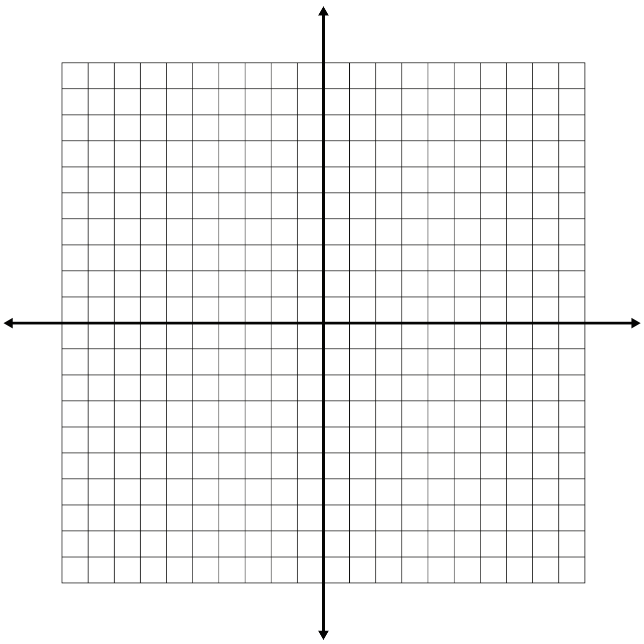


Example 11

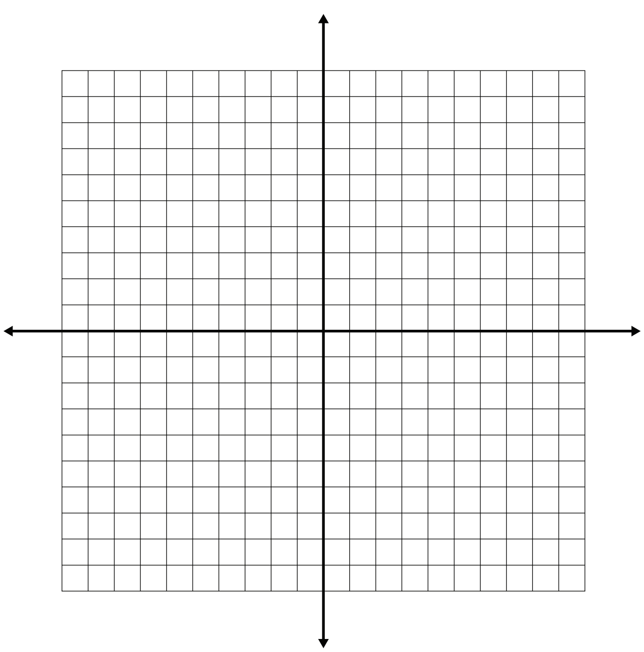
Answer the following questions:

What Transformation Occured?

a) The graph of $y = x^2 + 3$ is vertically translated so it passes through the point (2, 10). Write the equation of the applied transformation. *Solve graphically first, then solve algebraically.*



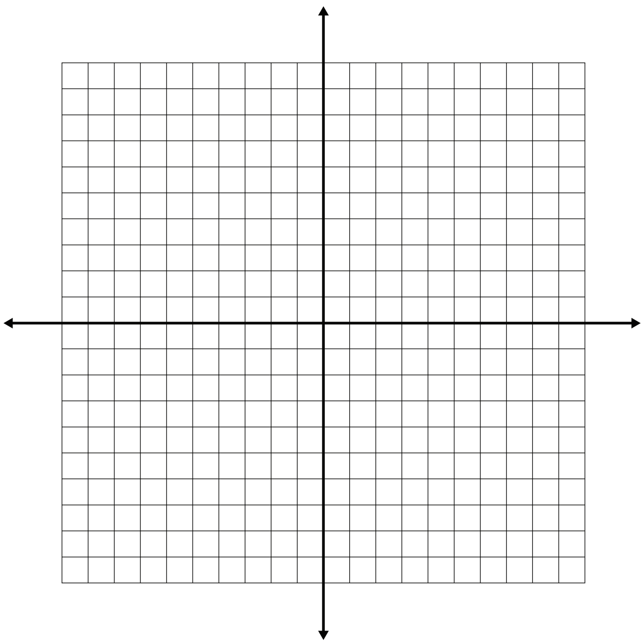
b) The graph of $y = (x + 2)^2$ is horizontally translated so it passes through the point (6, 9). Write the equation of the applied transformation.
Solve graphically first, then solve algebraically.



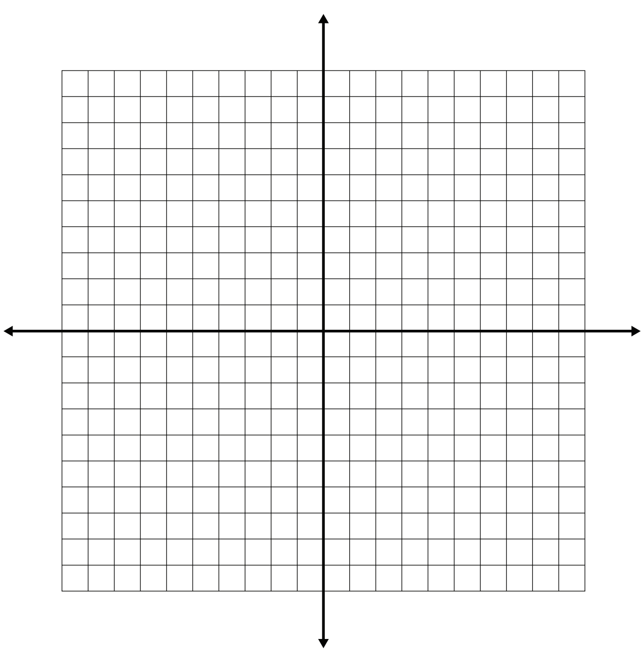
Example 12 Answer the following questions:

What Transformation Occured?

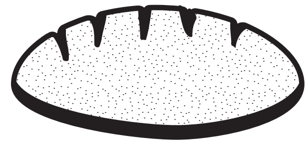
a) The graph of $y = x^2 - 2$ is vertically stretched so it passes through the point (2, 6). Write the equation of the applied transformation. *Solve graphically first, then solve algebraically.*



b) The graph of $y = (x - 1)^2$ is transformed by the equation $y = f(bx)$. The transformed graph passes through the point (-4, 4). Write the equation of the applied transformation. *Solve graphically first, then solve algebraically.*

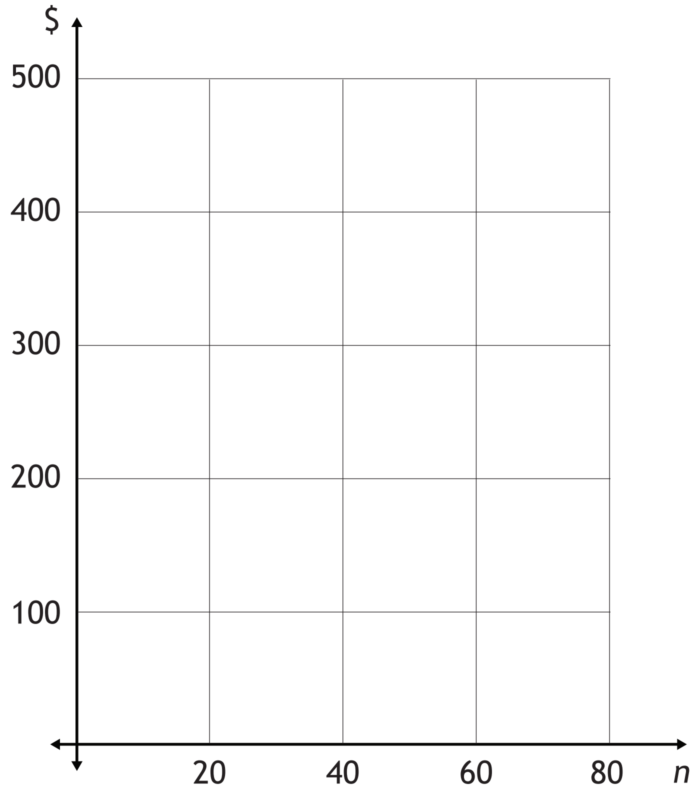


Example 13



Sam sells bread at a farmers' market for $5.00 per loaf. It costs $150 to rent a table for one day at the farmers' market, and each loaf of bread costs $2.00 to produce.

a) Write two functions, R(n) and C(n), to represent Sam's revenue and costs. Graph each function.

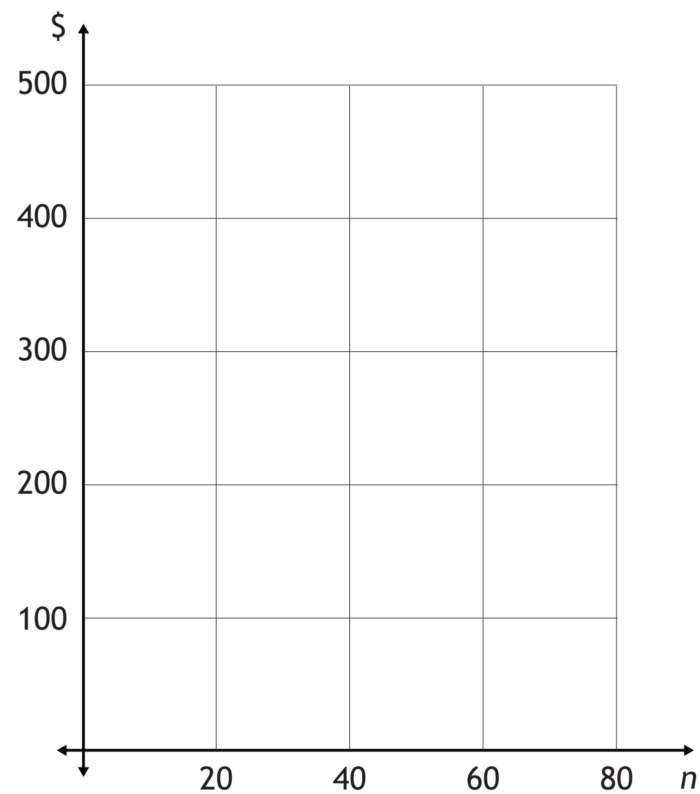


b) How many loaves of bread does Sam need to sell in order to make a profit?

c) The farmers' market raises the cost of renting a table by \$50 per day. Use a transformation to find the new cost function, $C_2(n)$.

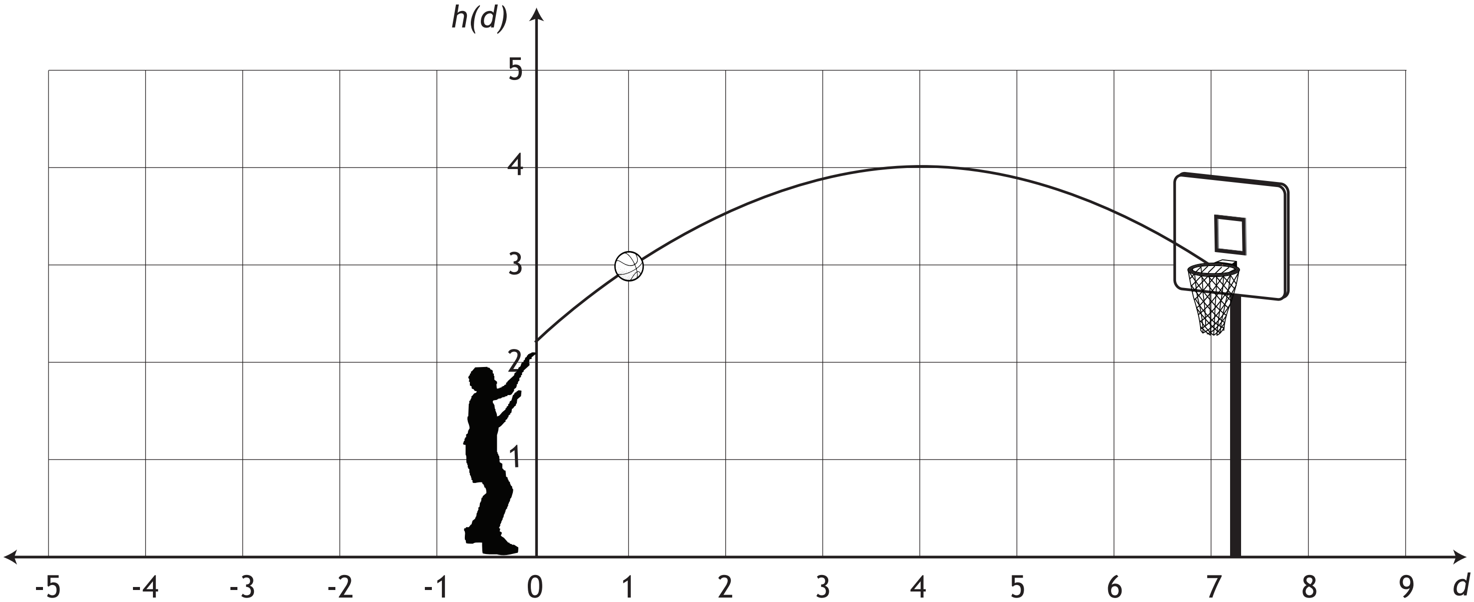
d) In order to compensate for the increase in rental costs, Sam will increase the price of a loaf of bread by 20%. Use a transformation to find the new revenue function, $R_2(n)$.

e) Draw the transformed functions from parts (c) and (d). How many loaves of bread does Sam need to sell now in order to break even?



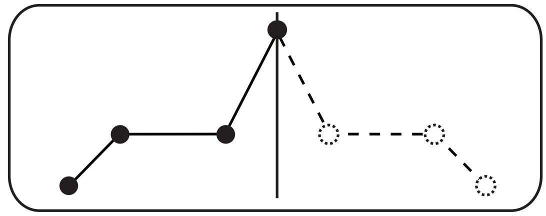
Example 14

A basketball player throws a basketball. The path can be modeled with $h(d) = -\frac{1}{9}(d - 4)^2 + 4$.



a) Suppose the player moves 2 m closer to the hoop before making the shot. Determine the equation of the transformed graph, draw the graph, and predict the outcome of the shot.

b) If the player moves so the equation of the shot is $h(d) = -\frac{1}{9}(d + 1)^2 + 4$, what is the horizontal distance from the player to the hoop?



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Example 1

Combined Transformations

Combining Stretches and Reflections

a) Identify each parameter in the general transformation equation: $y = af[b(x - h)] + k$.

b) Describe the transformations in each equation:

i) $y = \frac{1}{3}f(5x)$

ii) $y = 2f(\frac{1}{4}x)$

iii) $y = -\frac{1}{2}f(\frac{1}{3}x)$

iv) $y = -3f(-2x)$

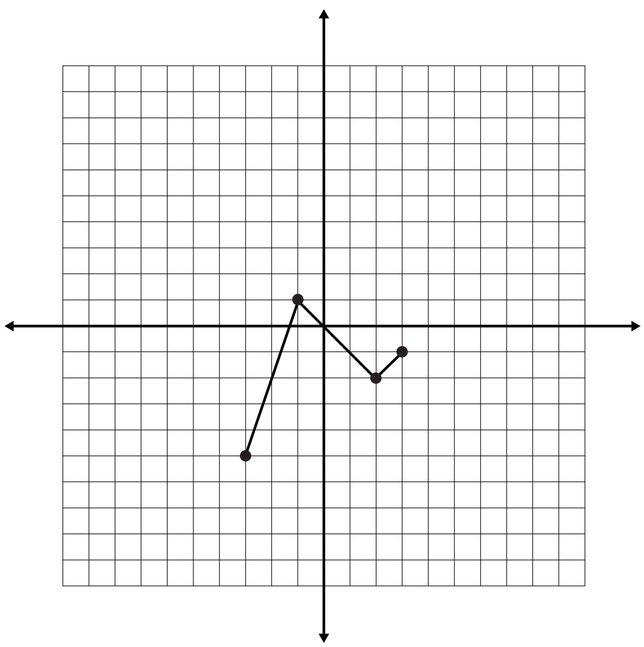
Transformations and Operations
LESSON TWO - *Combined Transformations*
Lesson Notes

$$y = af[b(x - h)] + k$$

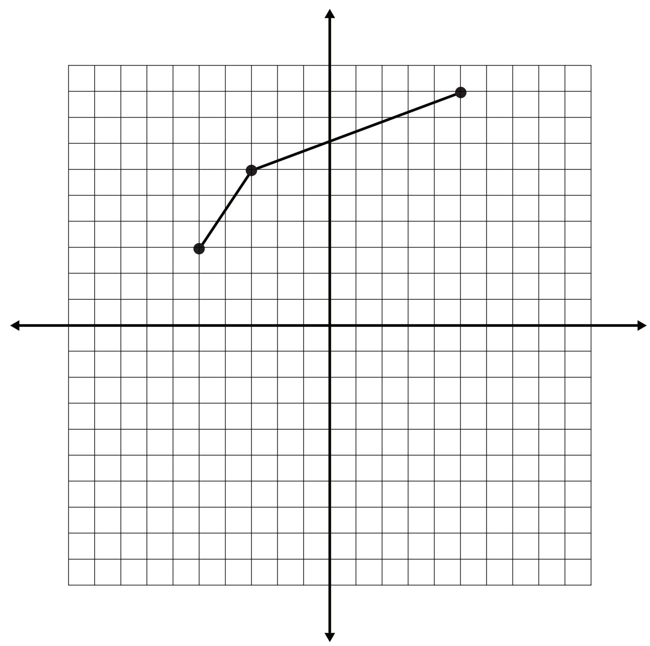
Example 2 Draw the transformation of each graph.

Combining Stretches and Reflections

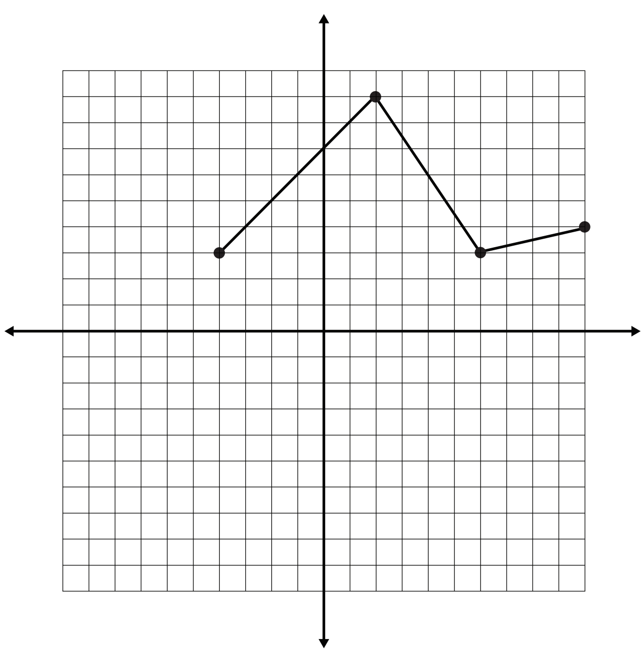
a) $y = 2f(\frac{1}{3}x)$



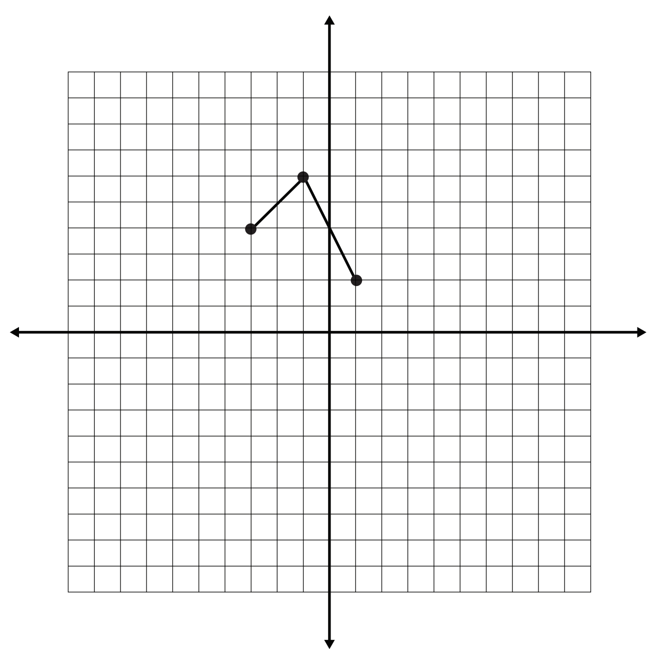
b) $y = \frac{1}{3}f(-x)$



c) $y = -f(2x)$



d) $y = -\frac{1}{2}f(-x)$



Example 3 Answer the following questions:

Combining Translations

a) Find the horizontal translation of $y = f(x + 3)$ using three different methods.

Opposite Method: Zero Method: Double Sign Method:

b) Describe the transformations in each equation:

i) $y = f(x - 1) + 3$

ii) $y = f(x + 2) - 4$

iii) $y = f(x - 2) - 3$

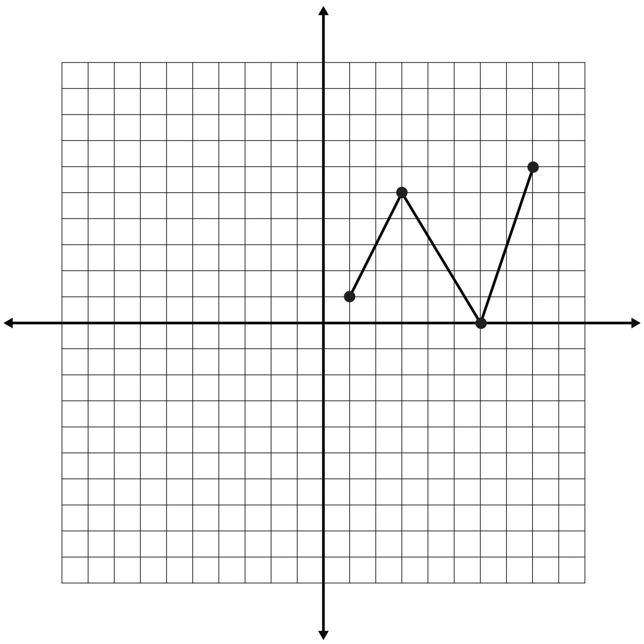
iv) $y = f(x + 7) + 5$

$y = af[b(x - h)] + k$

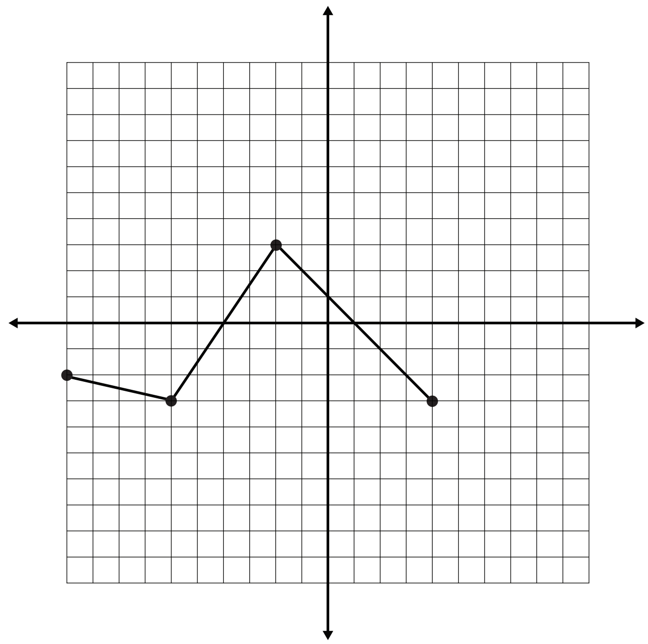
Example 4 Draw the transformation of each graph.

Combining Translations

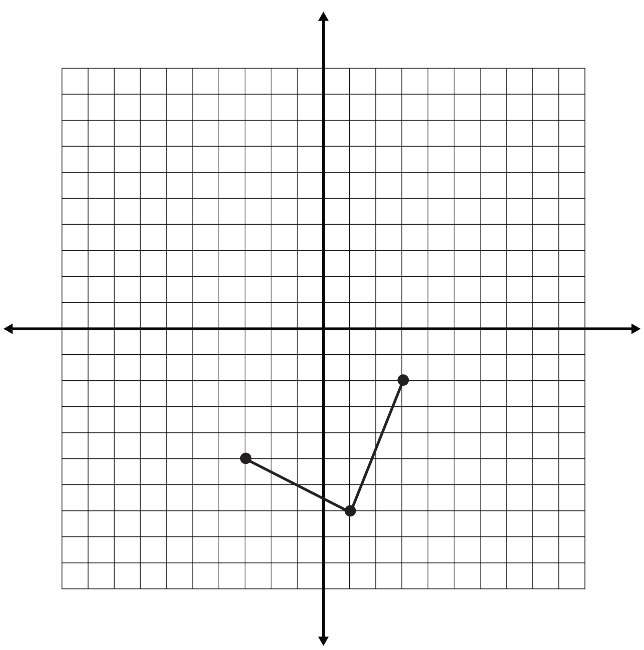
a) $y = f(x + 5) - 3$



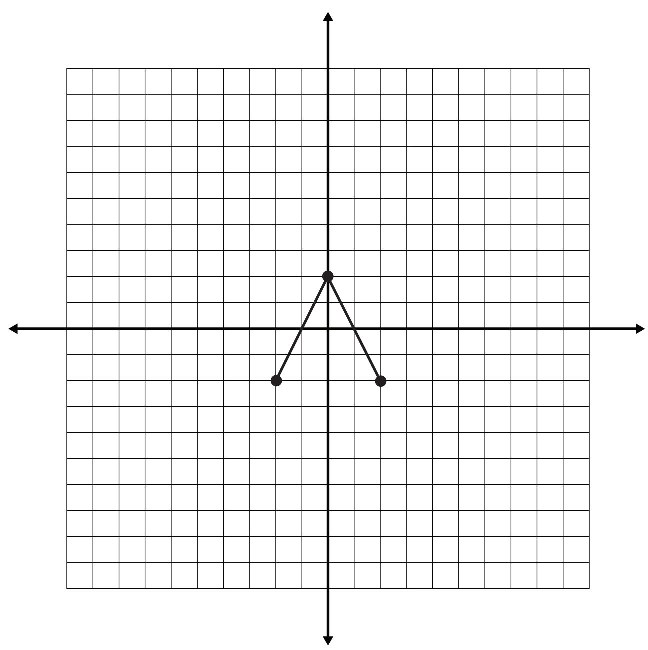
b) $y = f(x - 3) + 7$



c) $y - 12 = f(x - 6)$



d) $y + 2 = f(x + 8)$



Example 5 Answer the following questions:

Combining Stretches, Reflections, and Translations

a) When applying transformations to a graph, should they be applied in a specific order?

b) Describe the transformations in each equation.

i) $y = 2f(x + 3) + 1$

ii) $y = -f(\frac{1}{3}x) - 4$

iii) $y = \frac{1}{2}f[-(x + 2)] - 3$

iv) $y = -3f[-4(x - 1)] + 2$

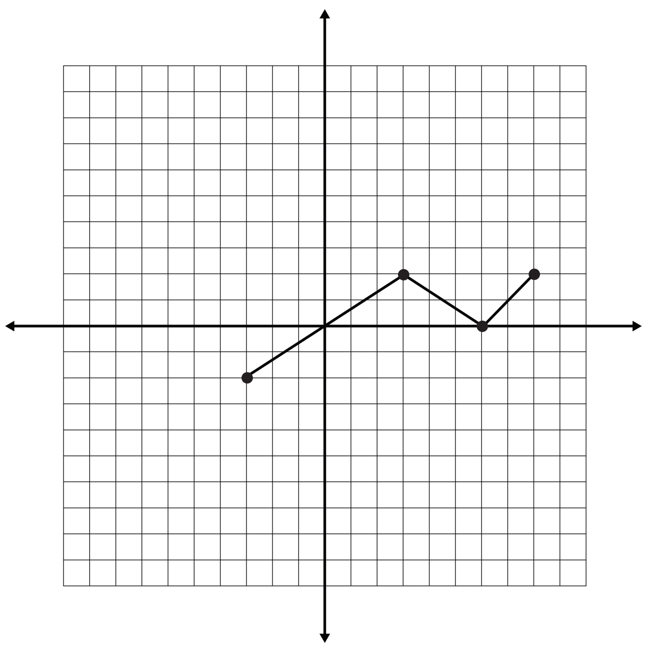
Transformations and Operations
LESSON TWO - *Combined Transformations*
Lesson Notes

$$y = af[b(x - h)] + k$$

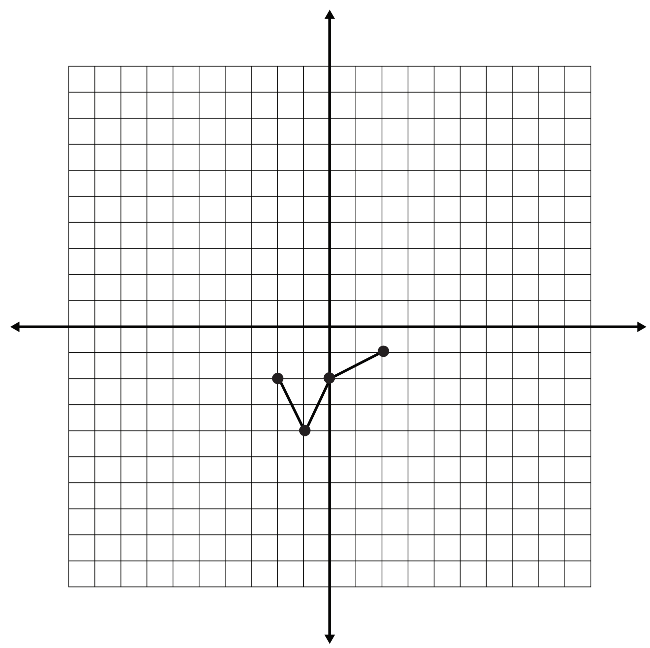
Example 6 Draw the transformation of each graph.

Combining Stretches, Reflections, and Translations

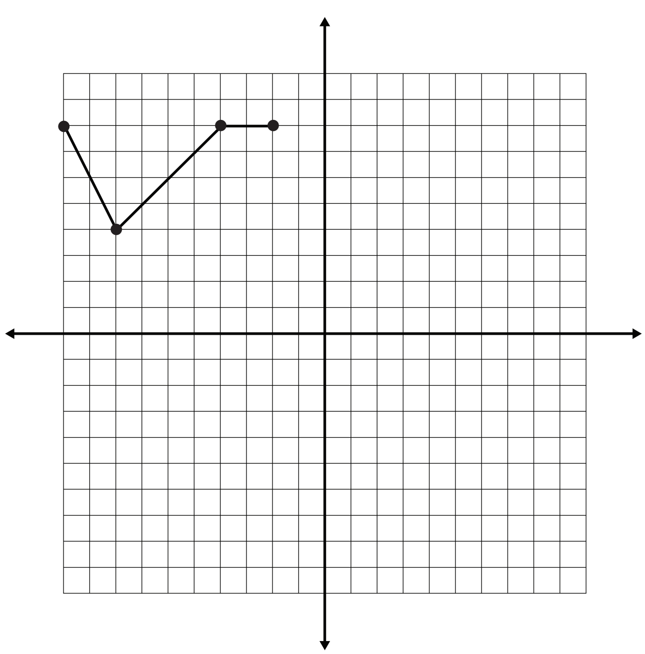
a) $y = -f(x) - 2$



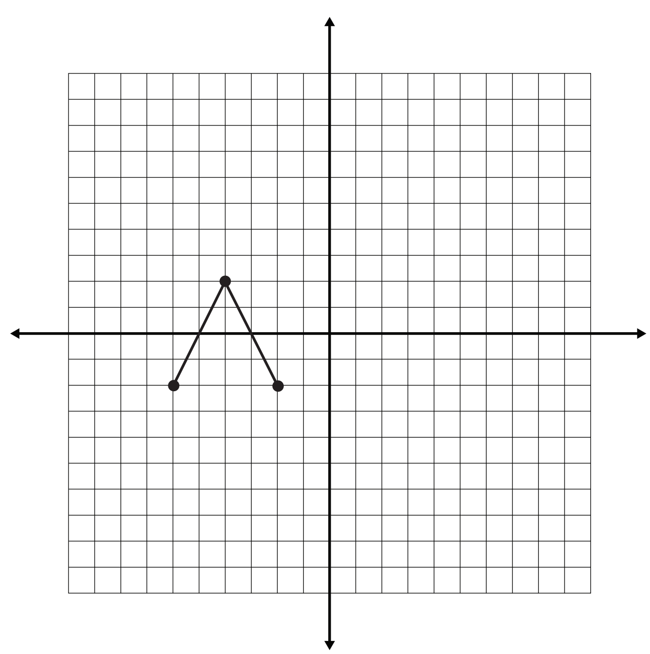
b) $y = f(-\frac{1}{4}x) + 1$



c) $y = -\frac{1}{4}f(2x) - 1$



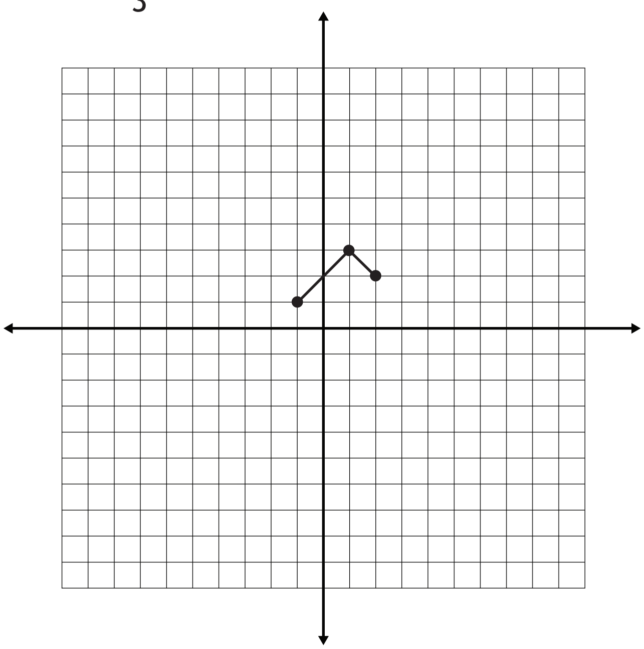
d) $2y - 8 = 6f(x - 2)$



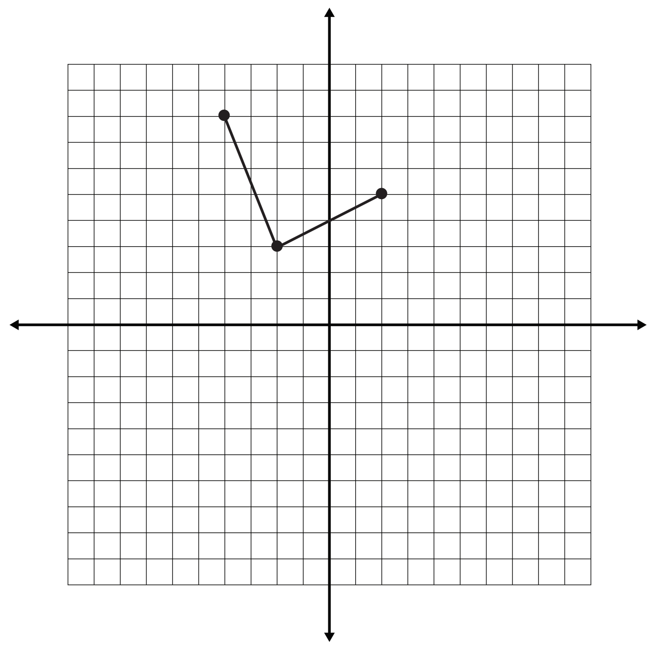
Example 7 Draw the transformation of each graph.

Combining Stretches, Reflections, and Translations ***(watch for b-factoring!)***

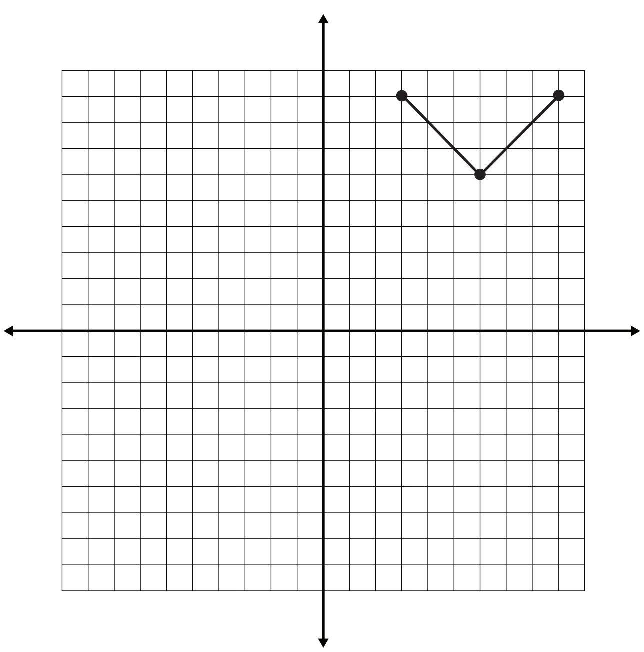
a) $y = f[\frac{1}{3}(x - 1)] + 1$



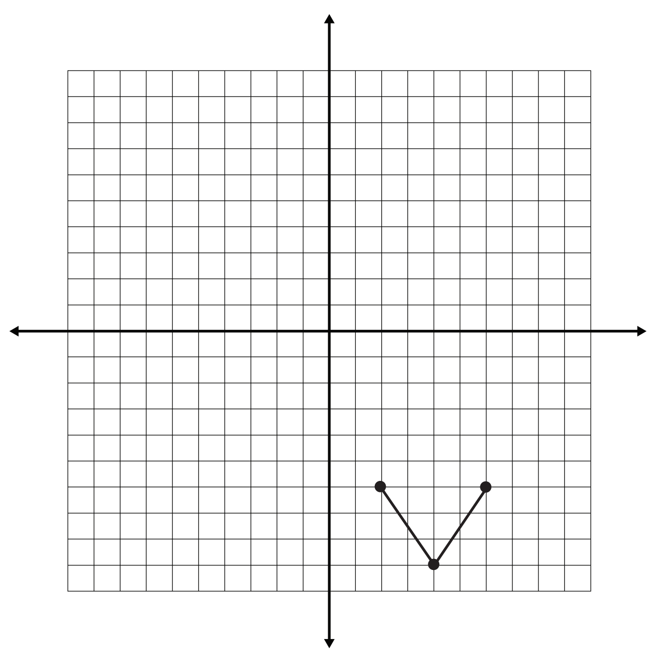
b) $y = f(2x + 6)$



c) $y = f(3x - 6) - 2$



d) $y = \frac{1}{3}f(-x - 4)$



$y = af[b(x - h)] + k$

Example 8 Answer the following questions:

Mappings

The mapping for combined transformations is:

$$(x, y) \rightarrow \left(\frac{x_i}{b} + h,\ ay_i + k\right)$$

a) If the point (2, 0) exists on the graph of y = f(x), find the coordinates of the new point after the transformation y = f(-2x + 4).

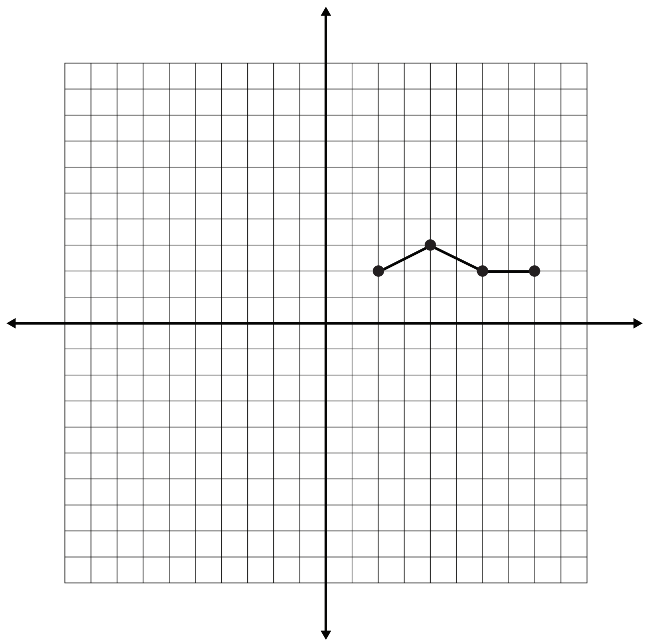
b) If the point (5, 4) exists on the graph of y = f(x), find the coordinates of the new point after the transformation $y = \frac{1}{2}f(5x - 10) + 4$.

c) The point (m, n) exists on the graph of y = f(x). If the transformation y = 2f(2x) + 5 is applied to the graph, the transformed point is (4, 7). Find the values of m and n.

Example 9 For each transformation description, write the transformation equation. Use mappings to draw the transformed graph.

Mappings

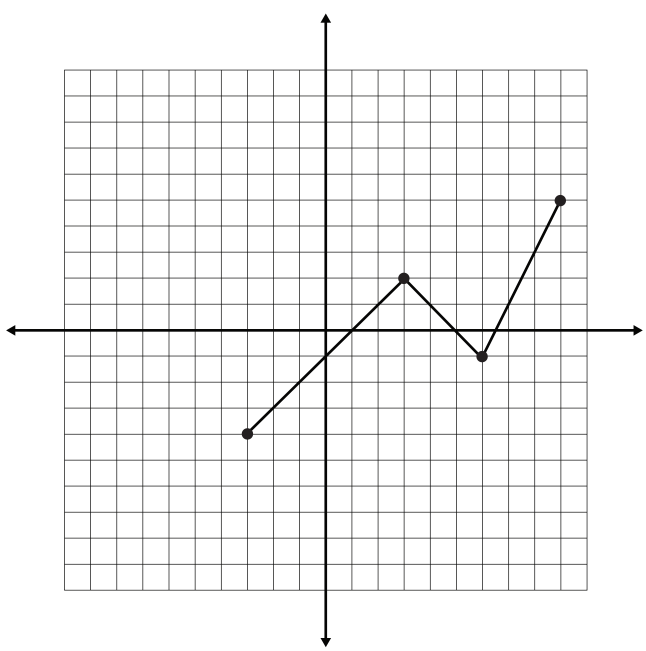
a) The graph of y = f(x) is vertically stretched by a factor of 3, reflected about the x-axis, and translated 2 units to the right.



Transformation Equation:

Mappings:

b) The graph of y = f(x) is horizontally stretched by a factor of $\frac{1}{3}$, reflected about the x-axis, and translated 2 units left.



Transformation Equation:

Mappings:

Transformations and Operations
LESSON TWO - *Combined Transformations*
Lesson Notes

$$y = af[b(x - h)] + k$$

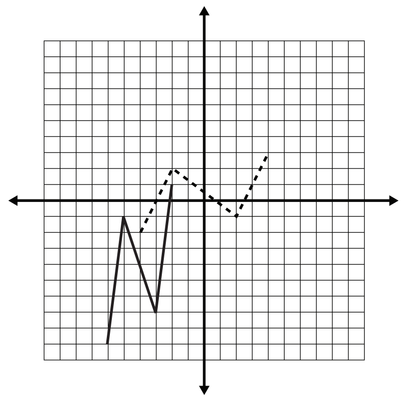
Example 10 Order of Transformations. Axis-Independence

Greg applies the transformation y = -2f[-2(x + 4)] - 3 to the graph below, using the transformation order rules learned in this lesson.

Greg's Transformation Order:

Stretches & Reflections:
1) Vertical stretch by a scale factor of 2
2) Reflection about the x-axis
3) Horizontal stretch by a scale factor of 1/2
4) Reflection about the y-axis

Translations:
5) Vertical translation 3 units down
6) Horizontal translation 4 units left



Original graph: - - - - - - - - - - - - - -

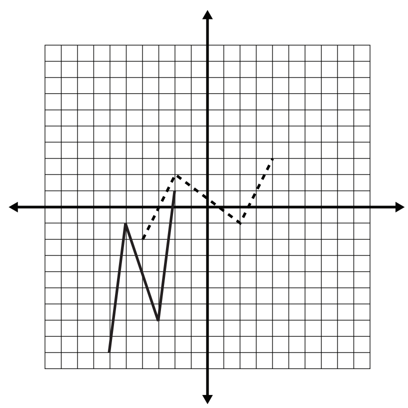
Transformed graph: ________

Next, Colin applies the same transformation, y = -2f[-2(x + 4)] - 3, to the graph below. He tries a different transformation order, applying all the vertical transformations first, followed by all the horizontal transformations.

Colin's Transformation Order:

Vertical Transformations:
1) Vertical stretch by a scale factor of 2
2) Reflection about the x-axis
3) Vertical translation 3 units down.

Horizontal Transformations:
4) Horizontal stretch by a scale factor of 1/2
5) Reflection about the y-axis
6) Horizontal translation 4 units left

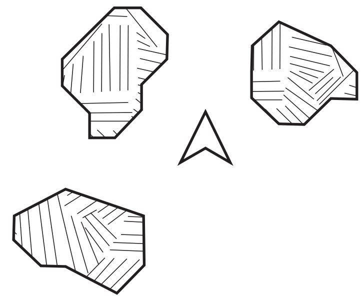


Original graph: - - - - - - - - - - - - - -

Transformed graph: ________

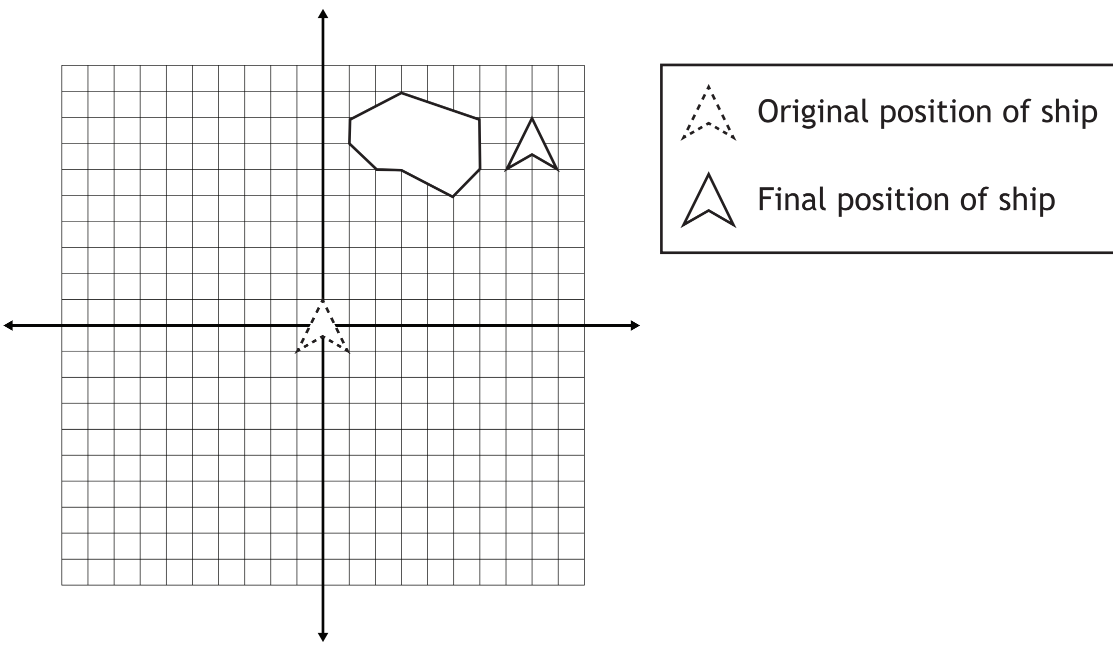
According to the transformation order rules we have been using in this lesson *(stretches & reflections first, translations last)*, Colin should obtain the wrong graph. However, Colin obtains the same graph as Greg! How is this possible?

Example 11

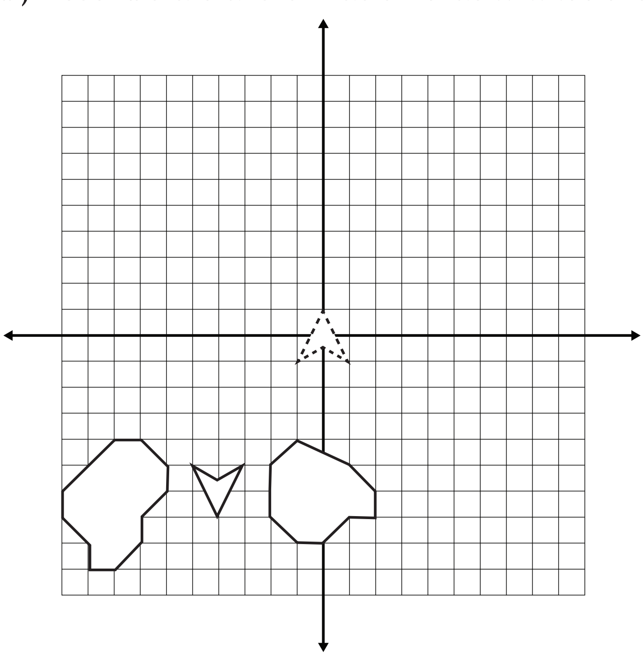


The goal of the video game *Space Rocks* is to pilot a spaceship through an asteroid field without colliding with any of the asteroids.

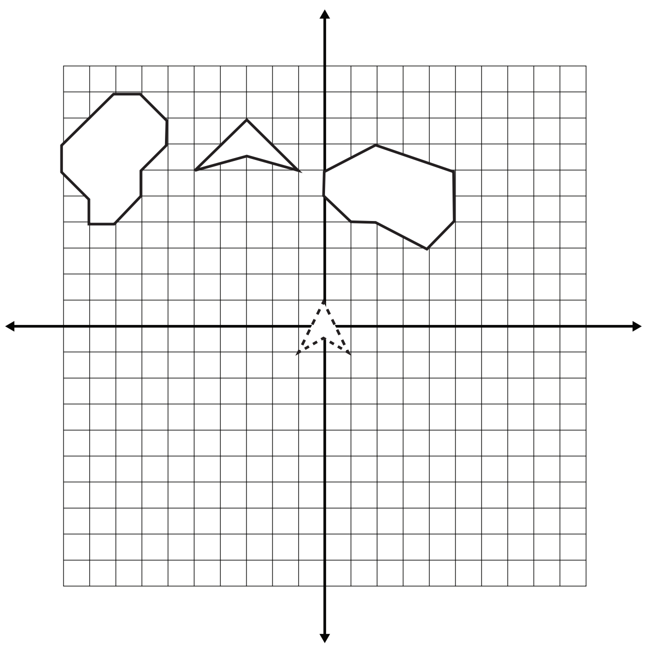
a) If the spaceship avoids the asteroid by navigating to the position shown, describe the transformation.



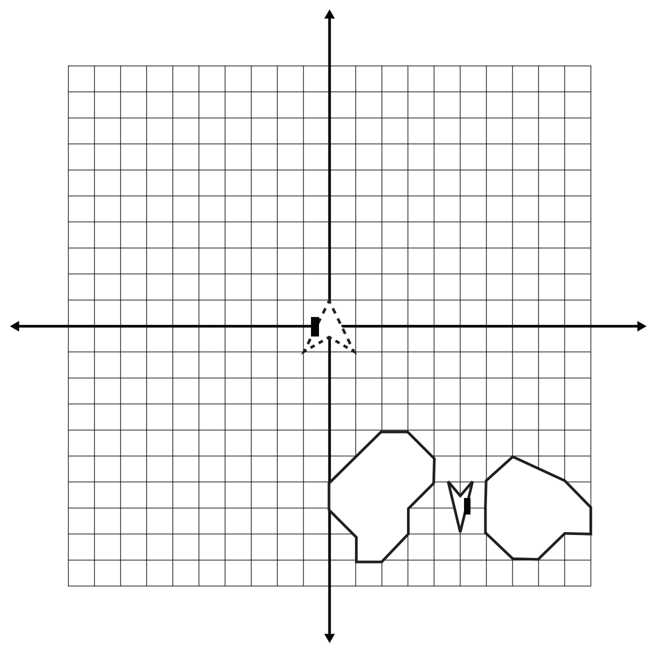
b) Describe a transformation that will let the spaceship pass through the asteroids.



c) The spaceship acquires a power-up that gives it greater speed, but at the same time doubles its width. What transformation is shown in the graph?



d) The spaceship acquires two power-ups. The first power-up halves the original width of the spaceship, making it easier to dodge asteroids. The second power-up is a left wing cannon. What transformation describes the spaceship's new size and position?

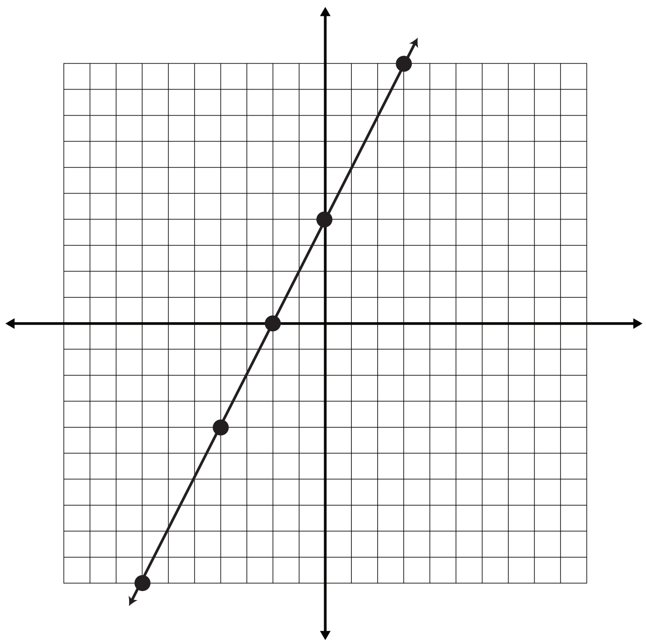


e) The transformations in parts (a - d) may **not** be written using y = af[b(x - h)] + k. Give two reasons why.

Example 1 Inverse Functions.

Finding an Inverse
(graphically and algebraically)

a) Given the graph of y = 2x + 4, draw the graph of the inverse.
What is the equation of the line of symmetry?



Inverse Mapping: (x, y) ⟶ (y, x)

(-7, -10) ⟶

(-4, -4) ⟶

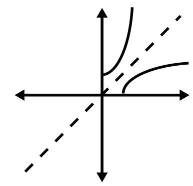
(-2, 0) ⟶

(0, 4) ⟶

(3, 10) ⟶

b) Find the inverse function algebraically.

Transformations and Operations
LESSON THREE - *Inverses*
Lesson Notes

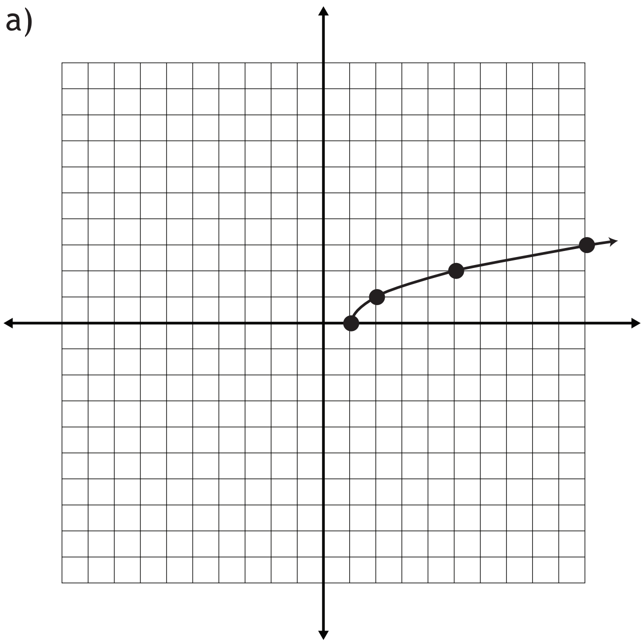


$f^{-1}(x)$

Example 2 For each graph, answer parts (i - iv).

Domain and Range

a)

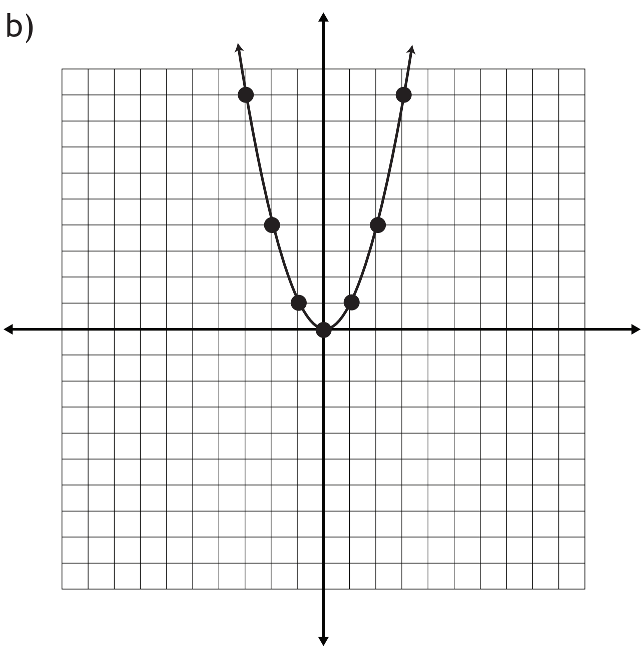


i) Draw the graph of the inverse.
ii) State the domain and range of the original graph.

iii) State the domain and range of the inverse graph.

iv) Can the inverse be represented with $f^{-1}(x)$?

b)

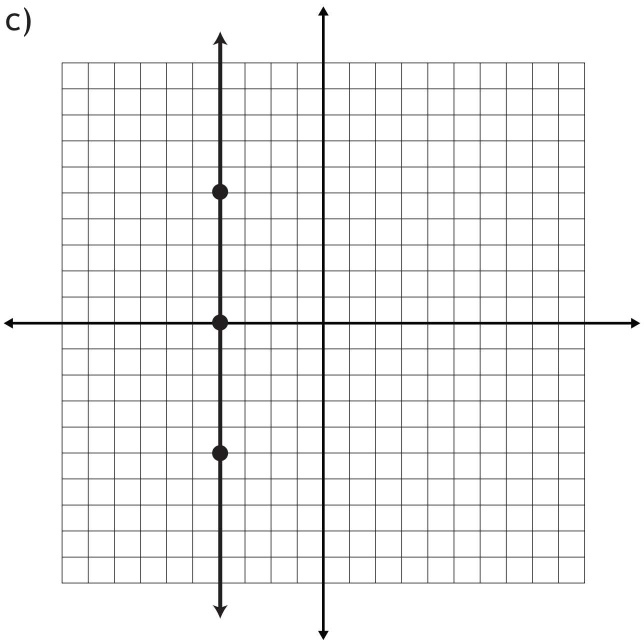


i) Draw the graph of the inverse.
ii) State the domain and range of the original graph.

iii) State the domain and range of the inverse graph.

iv) Can the inverse be represented with $f^{-1}(x)$?

c)

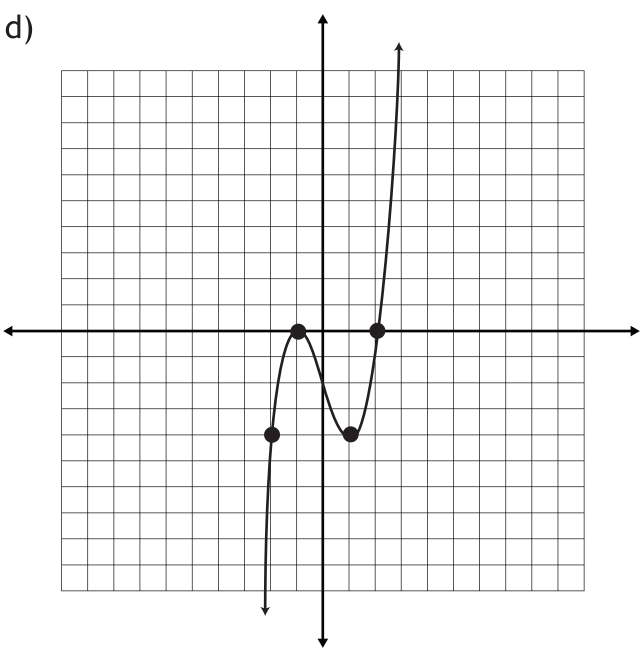


i) Draw the graph of the inverse.
ii) State the domain and range of the original graph.

iii) State the domain and range of the inverse graph.

iv) Can the inverse be represented with $f^{-1}(x)$?

d)



i) Draw the graph of the inverse.
ii) State the domain and range of the original graph.

iii) State the domain and range of the inverse graph.

iv) Can the inverse be represented with $f^{-1}(x)$?

Transformations and Operations
LESSON THREE - *Inverses*
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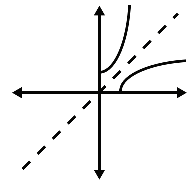


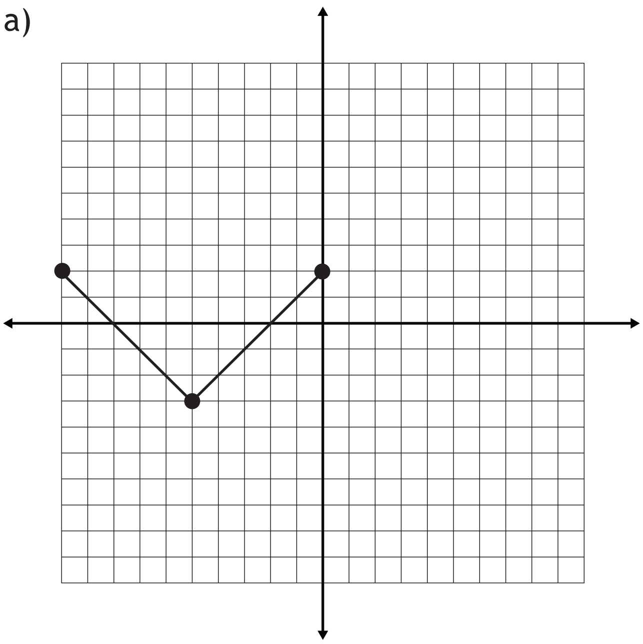
$f^{-1}(x)$

Example 3

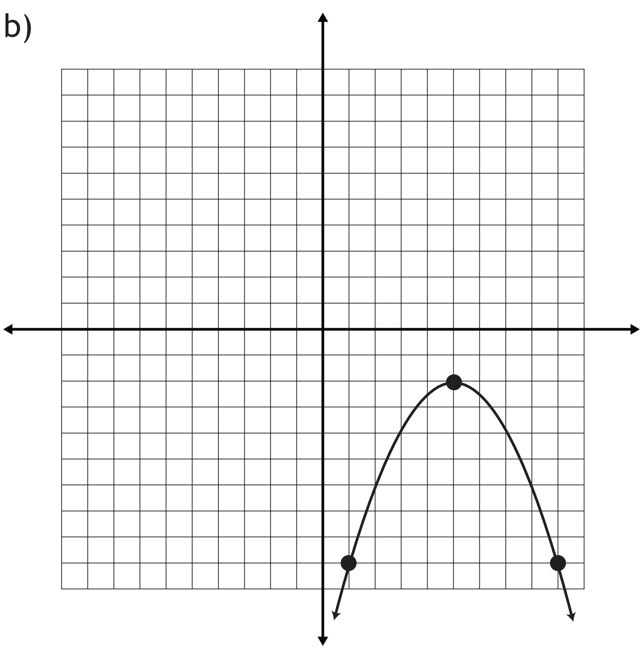
For each graph, draw the inverse. How should the domain of the original graph be restricted so the inverse is a function?

Domain Restrictions

a)



b)

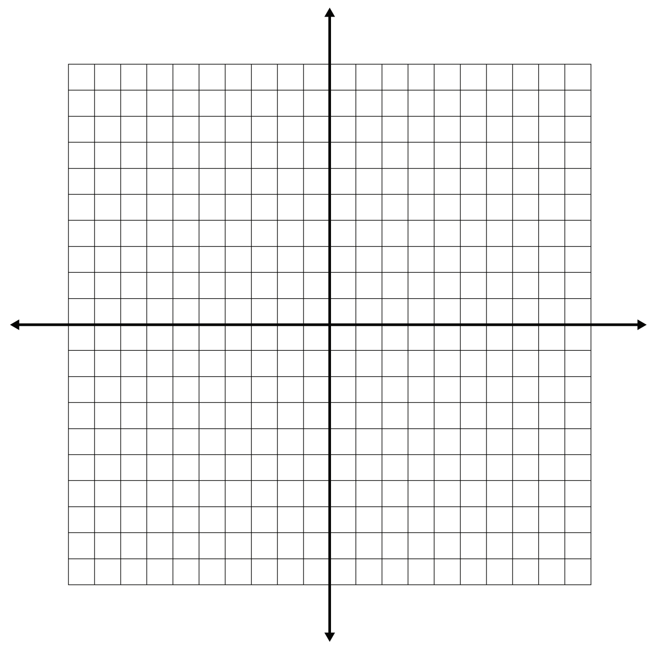


Example 4

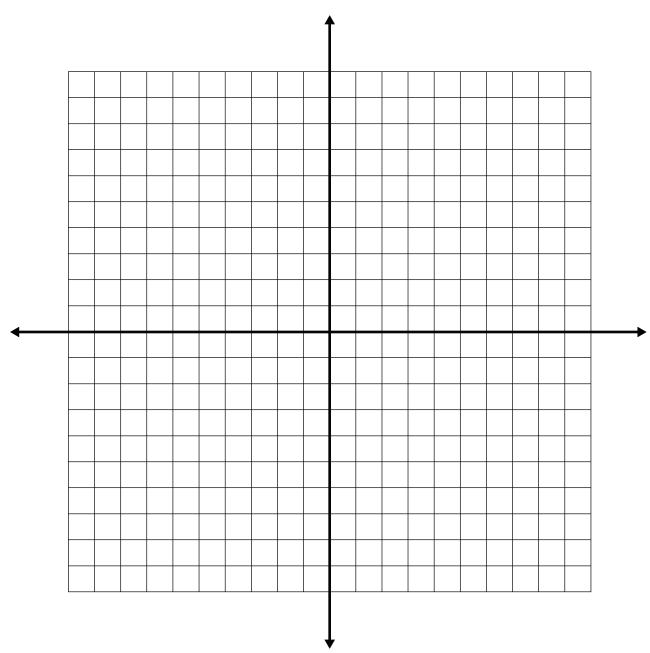
Find the inverse of each linear function algebraically. Draw the graph of the original function and the inverse. State the domain and range of both f(x) and its inverse.

Inverses of Linear Functions

a) $f(x) = x - 3$



b) $f(x) = -\frac{1}{2}x - 4$



Transformations and Operations
LESSON THREE - *Inverses*
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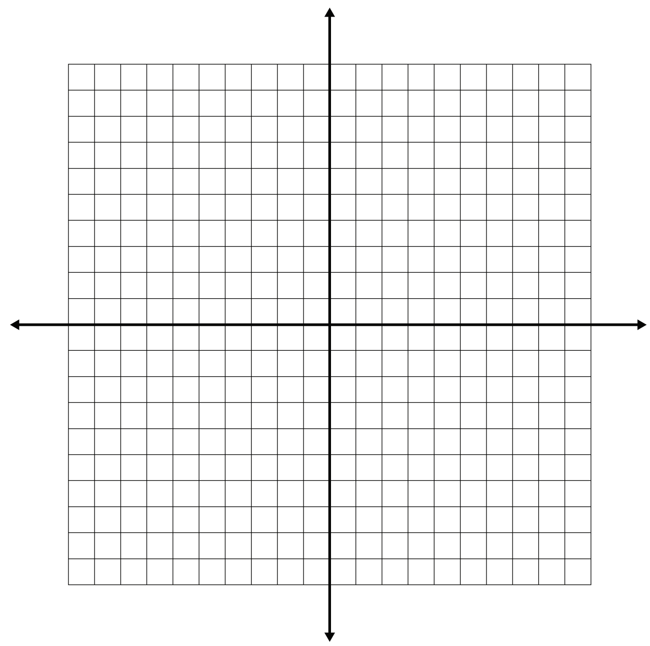
$f^{-1}(x)$

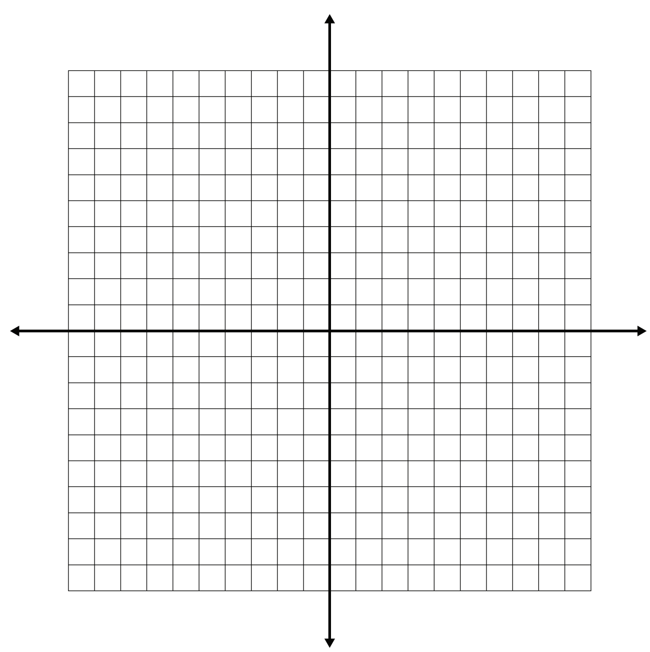
Example 5 Find the inverse of each quadratic function algebraically. Draw the graph of the original function and the inverse. Restrict the domain of f(x) so the inverse is a function.

Inverses of Quadratic Functions

a) $f(x) = x^2 - 4$



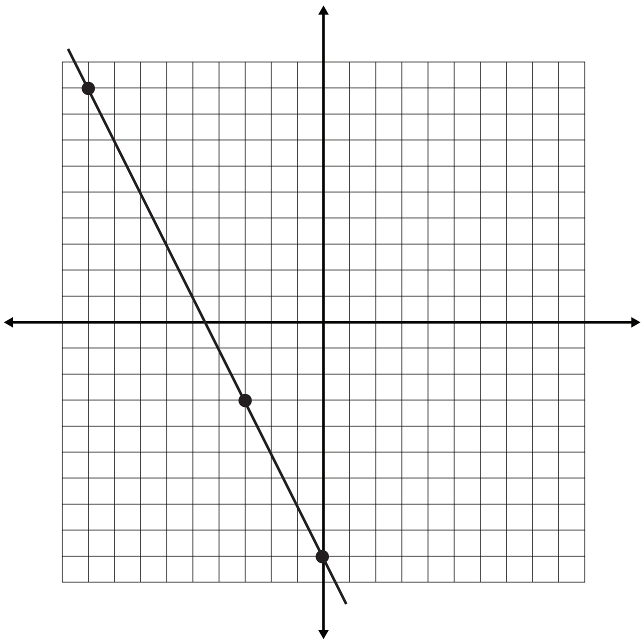
b) $f(x) = -(x + 3)^2 + 1$



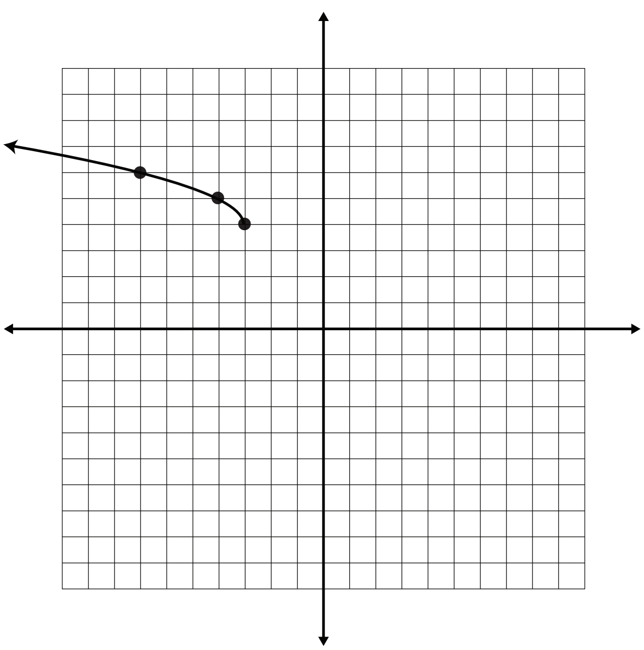
Example 6 For each graph, find the equation of the inverse.

Finding an Inverse From a Graph

a)



b)



Transformations and Operations
LESSON THREE - *Inverses*
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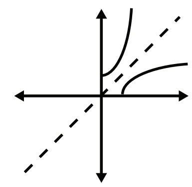
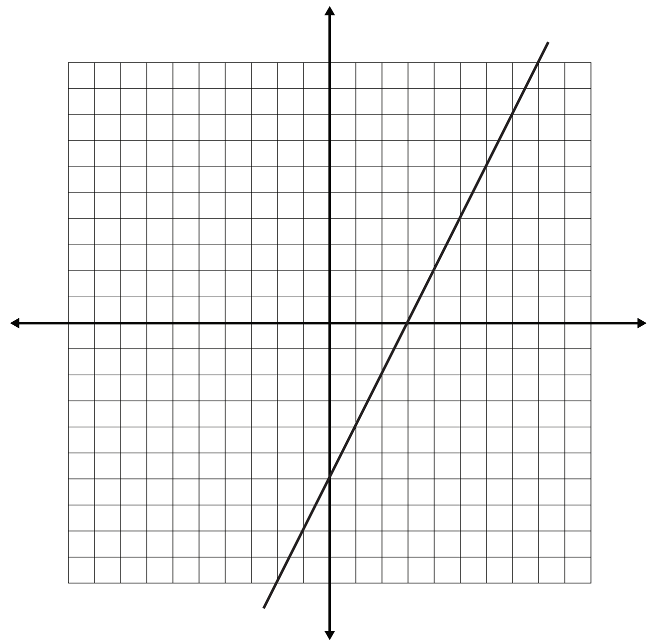


$f^{-1}(x)$

Example 7 Answer the following questions.

Understanding Inverse Function Notation

a) If $f(x) = 2x - 6$, find the inverse function and determine the value of $f^{-1}(10)$.



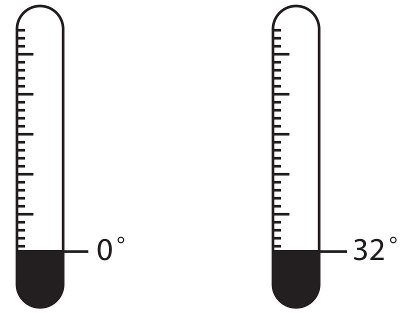
b) Given that $f(x)$ has an inverse function $f^{-1}(x)$, is it true that if $f(a) = b$, then $f^{-1}(b) = a$?

c) If $f^{-1}(4) = 5$, determine $f(5)$.

d) If $f^{-1}(k) = 18$, determine the value of k.

Example 8

In the Celsius temperature scale, the freezing point of water is set at 0 degrees. In the Fahrenheit temperature scale, 32 degrees is the freezing point of water. The formula to convert degrees Celsius to degrees Fahrenheit is: $F(C) = \frac{9}{5}C + 32$



Celsius Thermometer

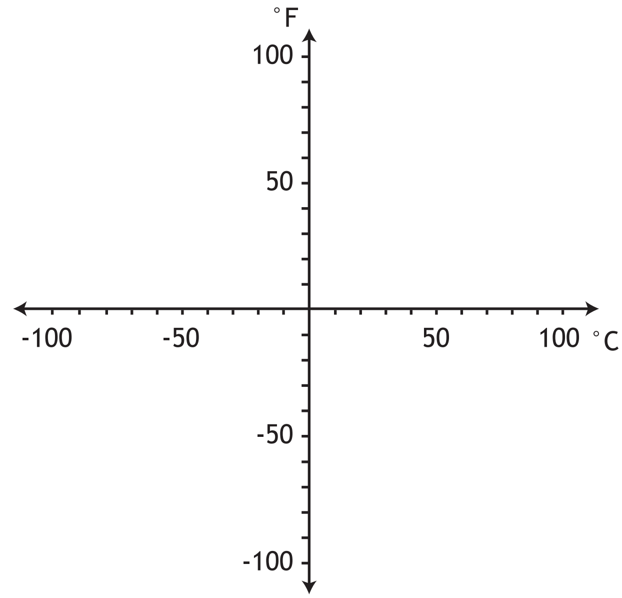
Fahrenheit Thermometer

a) Determine the temperature in degrees Fahrenheit for 28 °C.

b) Derive a function, C(F), to convert degrees Fahrenheit to degrees Celsius. Does one need to understand the concept of an inverse to accomplish this?

c) Use the function C(F) from part (b) to determine the temperature in degrees Celsius for 100 °F.

d) What difficulties arise when you try to graph F(C) and C(F) on the same grid?



e) Derive $F^{-1}(C)$. How does $F^{-1}(C)$ fix the graphing problem in part (d)?

f) Graph F(C) and $F^{-1}(C)$ using the graph above. What does the invariant point for these two graphs represent?

$(f + g)(x)$ $(f - g)(x)$

$(f \bullet g)(x)$ $\left(\frac{f}{g}\right)(x)$

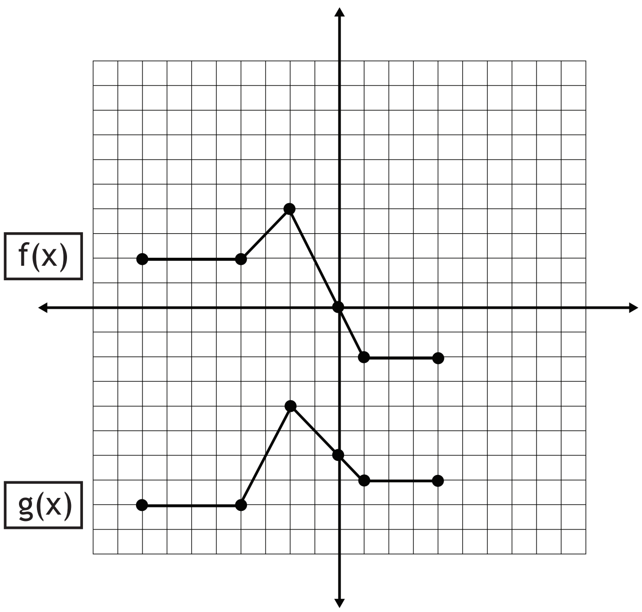
Transformations and Operations
LESSON FOUR - *Function Operations*
Lesson Notes

Example 1

Given the functions f(x) and g(x), complete the table of values for each operation and draw the graph. State the domain and range of the combined function.

Function Operations *(with a table of values)*

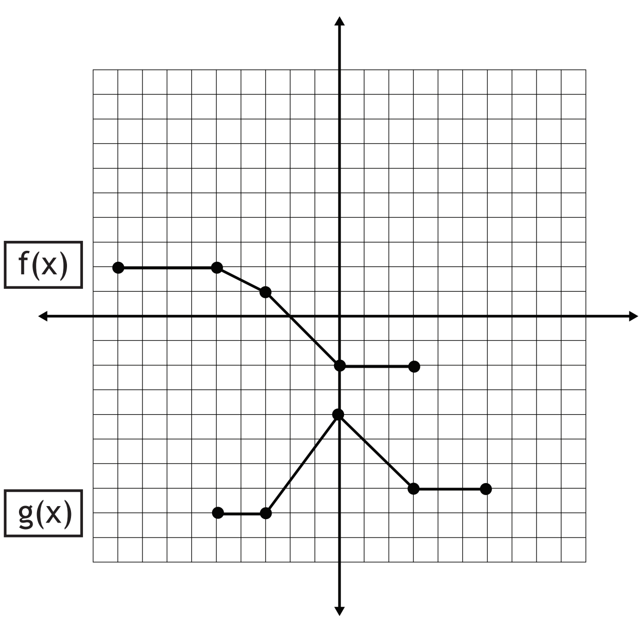
a) h(x) = (f + g)(x) *same as f(x) + g(x)*



x	(f + g)(x)
-8	
-4	
-2	
0	
1	
4	

Domain & Range:

b) h(x) = (f - g)(x) *same as f(x) - g(x)*



x	(f - g)(x)
-9	
-5	
-3	
0	
3	
6	

Domain & Range:

Set-Builder Notation

A ***set*** is simply a collection of numbers, such as {1, 4, 5}. We use ***set-builder notation*** to outline the rules governing members of a set.

$\{x \mid x \in R, x \geq -1\}$

State the variable. List conditions on the variable.

In words: "The variable is x, such that x can be any real number with the condition that x ≥ -1".

As a shortcut, set-builder notation can be reduced to just the most important condition.

$x \geq -1$

While this resource uses the shortcut for brevity, as set-builder notation is covered in previous courses, Math 30-1 students *are* expected to know how to read and write full set-builder notation.

Interval Notation

Math 30-1 students are expected to know that domain and range can be expressed using ***interval notation***.

() - Round Brackets: Exclude point from interval.
[] - Square Brackets: Include point in interval.
Infinity ∞ always gets a round bracket.

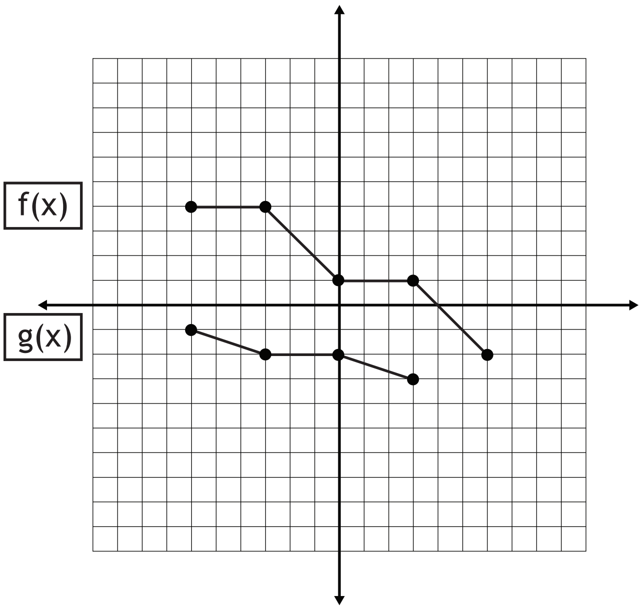
Examples: $x \geq -5$ becomes $[-5, \infty)$;
$1 < x \leq 4$ becomes $(1, 4]$;
$x \in R$ becomes $(-\infty, \infty)$;
$-8 \leq x < 2$ or $5 \leq x < 11$ becomes $[-8, 2) \cup [5, 11)$, where U means "or", or ***union of sets***;
$x \in R, x \neq 2$ becomes $(-\infty, 2) \cup (2, \infty)$;
$-1 \leq x \leq 3, x \neq 0$ becomes $[-1, 0) \cup (0, 3]$.

Transformations and Operations
LESSON FOUR - *Function Operations*
Lesson Notes

$(f + g)(x)$ $(f - g)(x)$
$(f \cdot g)(x)$ $\left(\frac{f}{g}\right)(x)$

Function Operations *(with a table of values)*

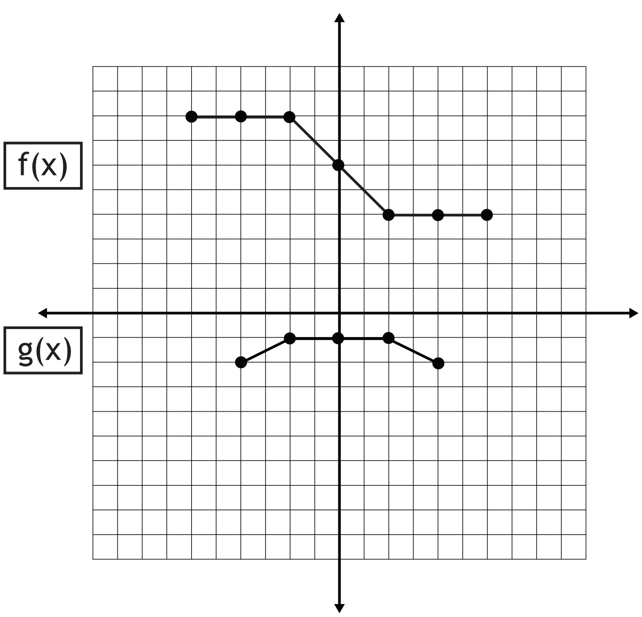
c) $h(x) = (f \cdot g)(x)$ *same as $f(x) \cdot g(x)$*



x	$(f \cdot g)(x)$
-6	
-3	
0	
3	
6	

Domain & Range:

d) $h(x) = \left(\frac{f}{g}\right)(x)$ *same as $f(x) \div g(x)$*



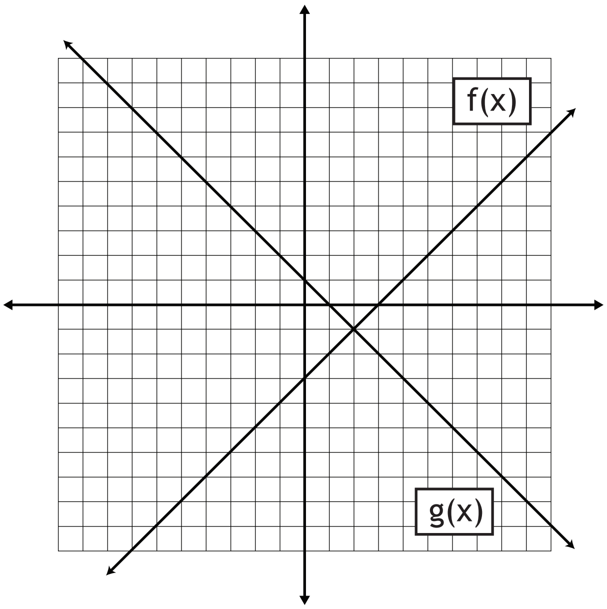
x	$(f \div g)(x)$
-6	
-4	
-2	
0	
2	
4	
6	

Domain & Range:

Example 2 Given the functions $f(x) = x - 3$ and $g(x) = -x + 1$, evaluate:

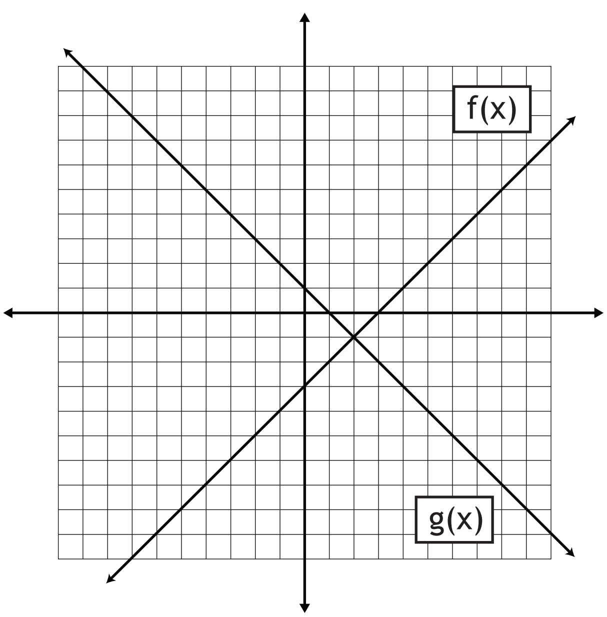
Function Operations
(graphically and algebraically)

a) $(f + g)(-4)$ *same as* $f(-4) + g(-4)$



i) using the graph	ii) using $h(x) = (f + g)(x)$

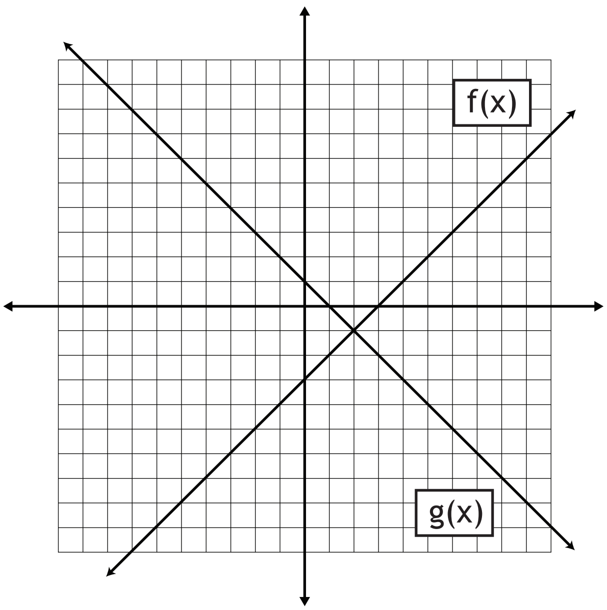
b) $(f - g)(6)$ *same as* $f(6) - g(6)$



i) using the graph	ii) using $h(x) = (f - g)(x)$

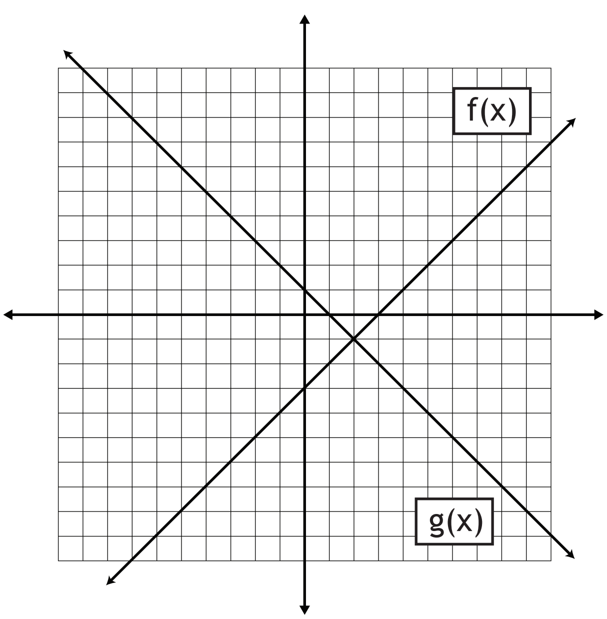
Function Operations
(graphically and algebraically)

c) $(fg)(-1)$ *same as* $f(-1) \bullet g(-1)$



i) using the graph	ii) using $h(x) = (f \bullet g)(x)$

d) $\left(\frac{f}{g}\right)(5)$ *same as* $f(5) \div g(5)$



i) using the graph	ii) using $h(x) = (f \div g)(x)$

$(f + g)(x)$ $(f - g)(x)$

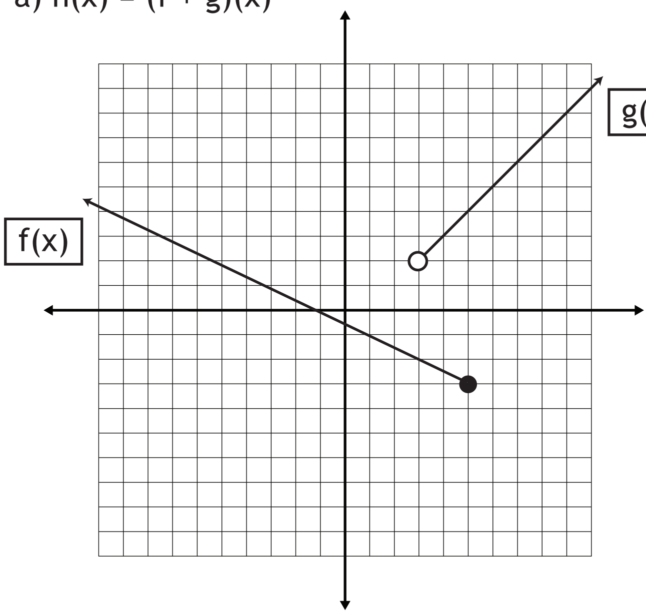
$(f \bullet g)(x)$ $\left(\frac{f}{g}\right)(x)$

Transformations and Operations
LESSON FOUR - *Function Operations*
Lesson Notes

Example 3 Draw each combined function and state the domain and range.

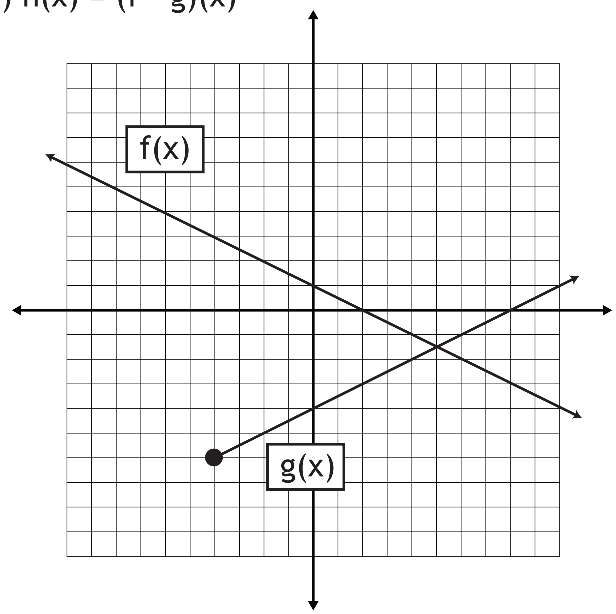
Combining Existing Graphs

a) h(x) = (f + g)(x)



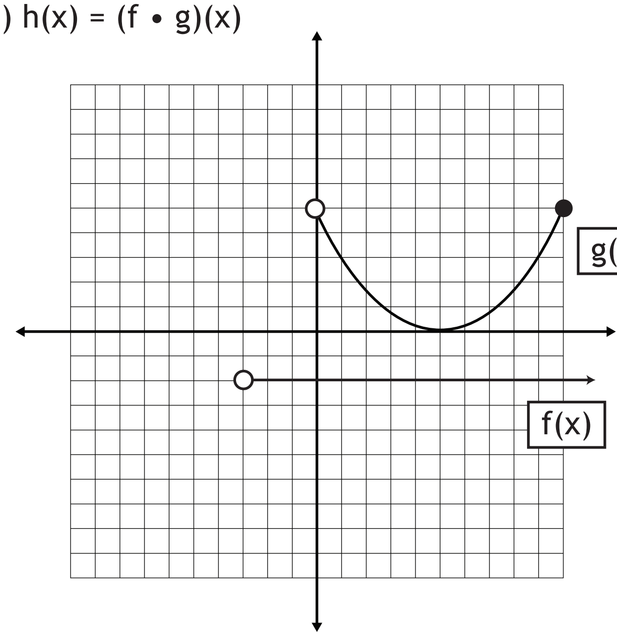
Domain & Range of h(x):

b) h(x) = (f - g)(x)



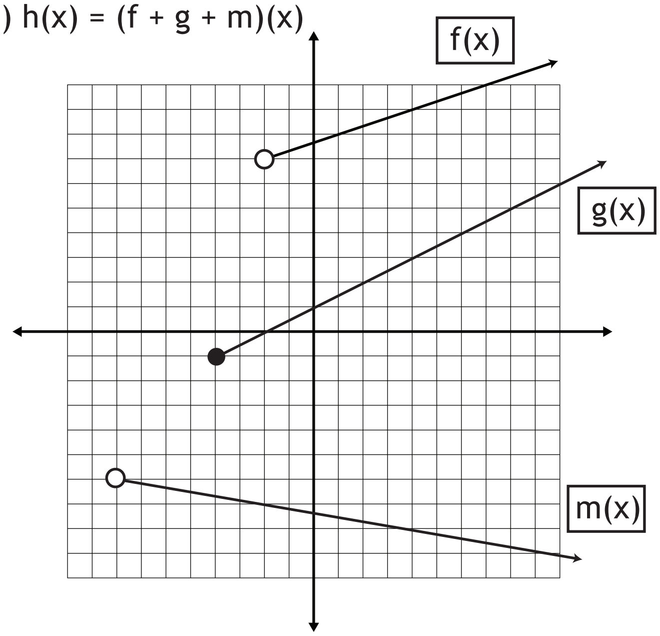
Domain & Range of h(x):

c) h(x) = (f • g)(x)



Domain & Range of h(x):

d) h(x) = (f + g + m)(x)



Domain & Range of h(x):

Transformations and Operations
LESSON FOUR - *Function Operations*
Lesson Notes

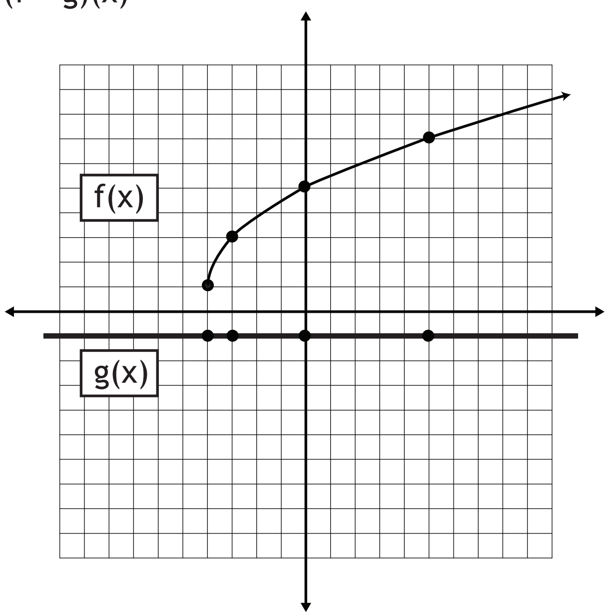
$(f + g)(x)$ **$(f - g)(x)$**

$(f \bullet g)(x)$ **$\left(\frac{f}{g}\right)(x)$**

Example 4 Given the functions $f(x) = 2\sqrt{x + 4} + 1$ and $g(x) = -1$, answer the following questions.

Function Operations
(with a radical function)

a) $(f + g)(x)$



i) Use a table of values to draw $(f + g)(x)$.

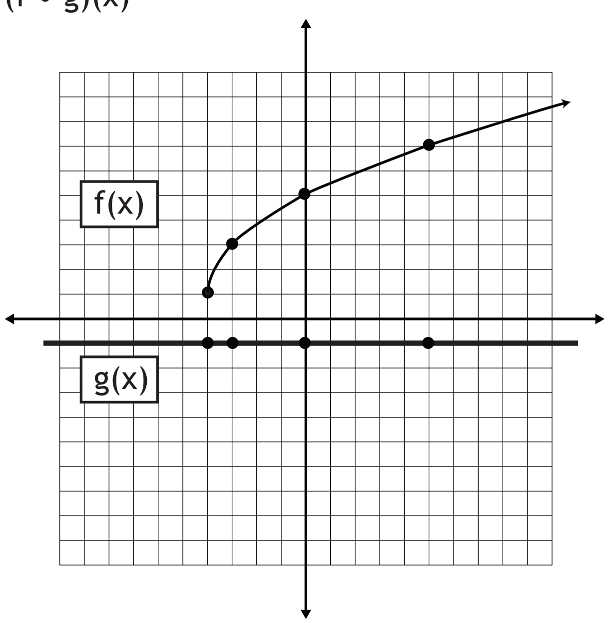
x	$(f + g)(x)$
-4	
-3	
0	
5	

ii) Derive $h(x) = (f + g)(x)$

iii) Domain & Range of $h(x)$

iv) Write a transformation equation that transforms the graph of $f(x)$ to $h(x)$.

b) $(f \bullet g)(x)$



i) Use a table of values to draw $(f \bullet g)(x)$.

x	$(f \bullet g)(x)$
-4	
-3	
0	
5	

ii) Derive $h(x) = (f \bullet g)(x)$

iii) Domain & Range of $h(x)$

iv) Write a transformation equation that transforms the graph of $f(x)$ to $h(x)$.

$(f + g)(x)$ $(f - g)(x)$

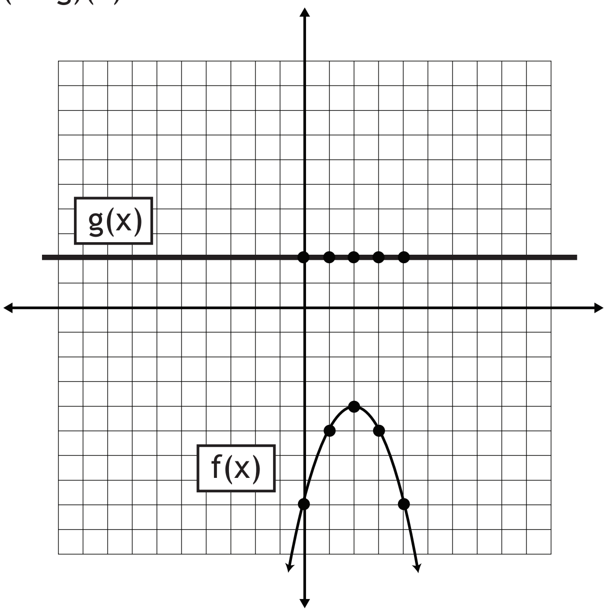
$(f \bullet g)(x)$ $\left(\frac{f}{g}\right)(x)$

Transformations and Operations
LESSON FOUR - *Function Operations*
Lesson Notes

Example 5 Given the functions $f(x) = -(x - 2)^2 - 4$ and $g(x) = 2$, answer the following questions.

Function Operations
(with a quadratic function)

a) $(f - g)(x)$



i) Use a table of values to draw $(f - g)(x)$.

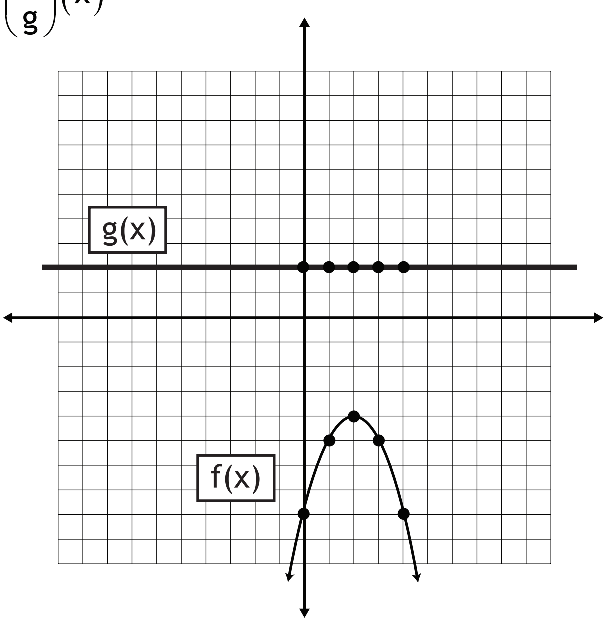
x	$(f - g)(x)$
0	
1	
2	
3	
4	

ii) Derive $h(x) = (f - g)(x)$

iii) Domain & Range of $h(x)$

iv) Write a transformation equation that transforms the graph of $f(x)$ to $h(x)$.

b) $\left(\frac{f}{g}\right)(x)$



i) Use a table of values to draw $(f \div g)(x)$.

x	$(f \div g)(x)$
0	
1	
2	
3	
4	

ii) Derive $h(x) = (f \div g)(x)$

iii) Domain & Range of $h(x)$

iv) Write a transformation equation that transforms the graph of $f(x)$ to $h(x)$.

Transformations and Operations
LESSON FOUR - *Function Operations*
Lesson Notes

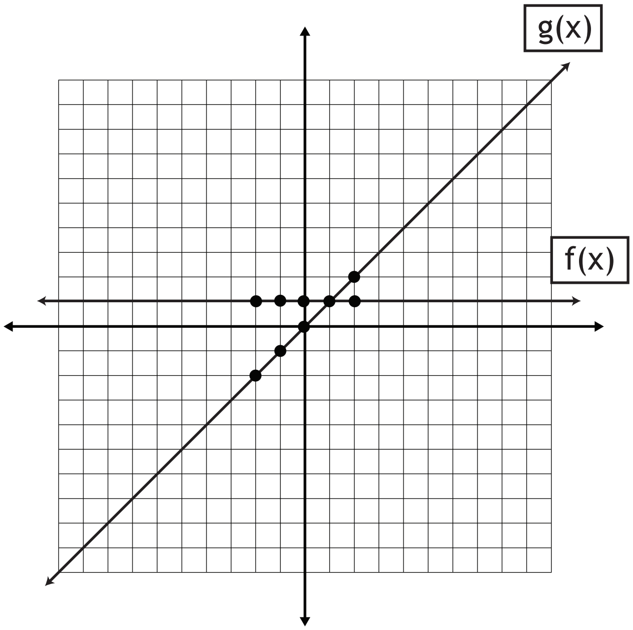
$(f + g)(x)$ $(f - g)(x)$

$(f \cdot g)(x)$ $\left(\frac{f}{g}\right)(x)$

Example 6 Draw the graph of $h(x) = \left(\frac{f}{g}\right)(x)$. Derive h(x) and state the domain and range.

Function Operations
(with a rational function)

a) f(x) = 1 and g(x) = x



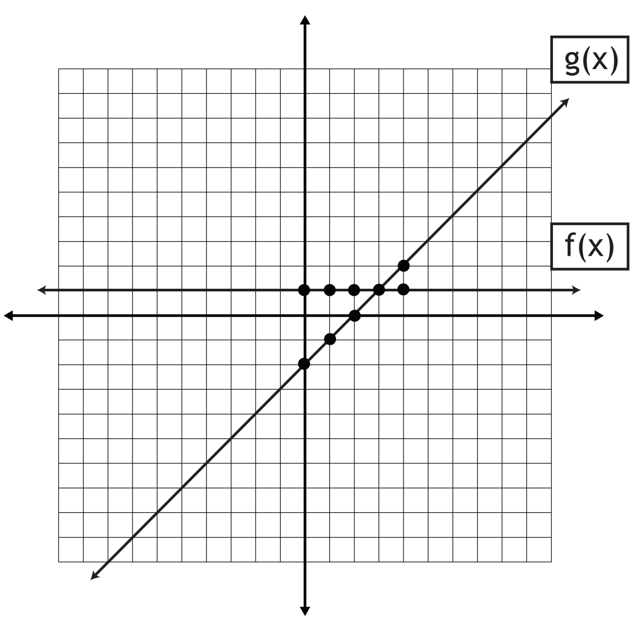
i) Use a table of values to draw (f ÷ g)(x).

x	*(f ÷ g)(x)*
-2	
-1	
0	
1	
2	

ii) Derive h(x) = (f ÷ g)(x)

iii) Domain & Range of h(x)

b) f(x) = 1 and g(x) = x - 2



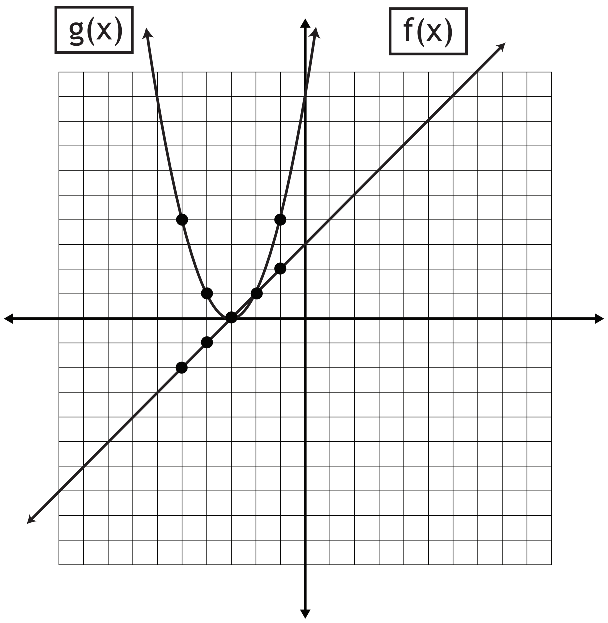
i) Use a table of values to draw (f ÷ g)(x).

x	*(f ÷ g)(x)*
0	
1	
2	
3	
4	

ii) Derive h(x) = (f ÷ g)(x)

iii) Domain & Range of h(x)

c) $f(x) = x + 3$ and $g(x) = x^2 + 6x + 9$



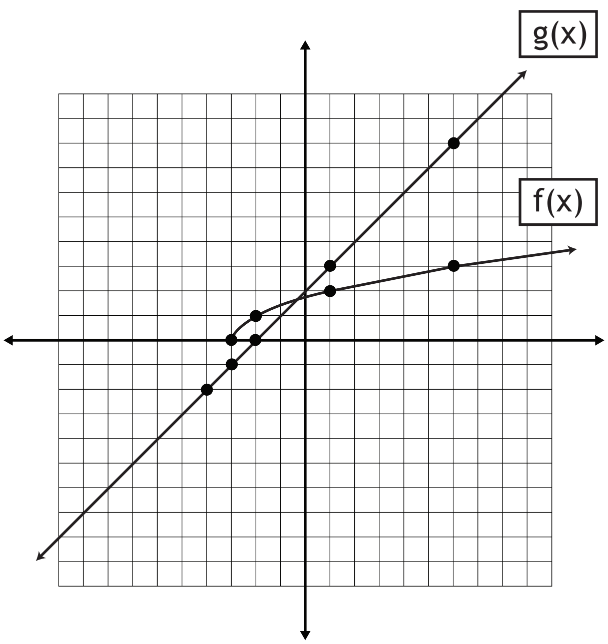
i) Use a table of values to draw $(f \div g)(x)$.

x	$(f \div g)(x)$
-5	
-4	
-3	
-2	
-1	

ii) Derive $h(x) = (f \div g)(x)$

iii) Domain & Range of $h(x)$

d) $f(x) = \sqrt{x + 3}$ and $g(x) = x + 2$



i) Use a table of values to draw $(f \div g)(x)$.

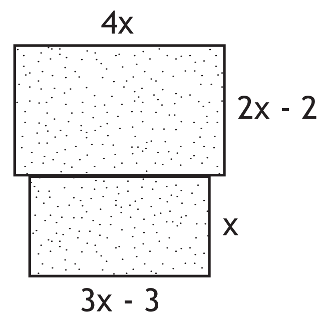
x	$(f \div g)(x)$
-4	
-3	
-2	
1	
6	

ii) Derive $h(x) = (f \div g)(x)$

iii) Domain & Range of $h(x)$

$(f + g)(x)$ $(f - g)(x)$ $(f \bullet g)(x)$ $\left(\frac{f}{g}\right)(x)$

Example 7 Two rectangular lots are adjacent to each other, as shown in the diagram.



a) Write a function, $A_L(x)$, for the area of the large lot.

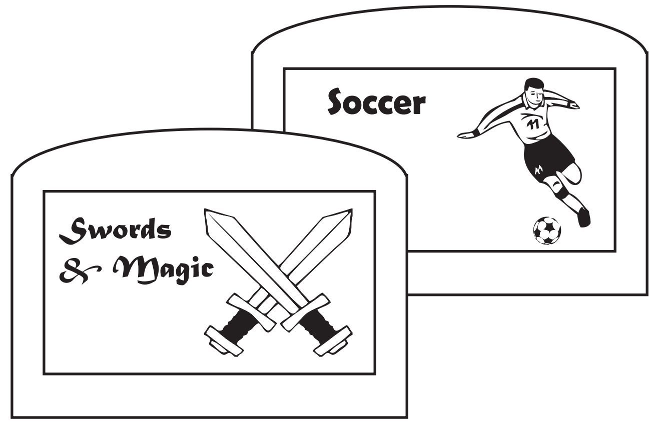
b) Write a function, $A_S(x)$, for the area of the small lot.

c) If the large rectangular lot is 10 m^2 larger than the small lot, use a function operation to solve for x.

d) Using a function operation, determine the total area of both lots.

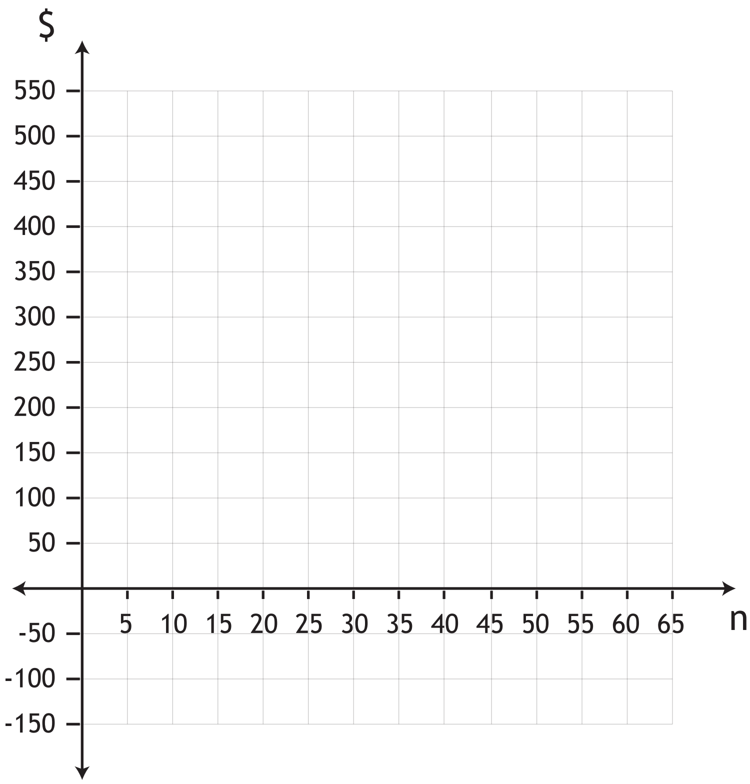
e) Using a function operation, determine how many times bigger the large lot is than the small lot.

Example 8



Greg wants to to rent a stand at a flea market to sell old video game cartridges. He plans to acquire games for \$4 each from an online auction site, then sell them for \$12 each. The cost of renting the stand is \$160 for the day.

a) Using function operations, derive functions for revenue *R(n)*, expenses *E(n)*, and profit *P(n)*. Graph each function.

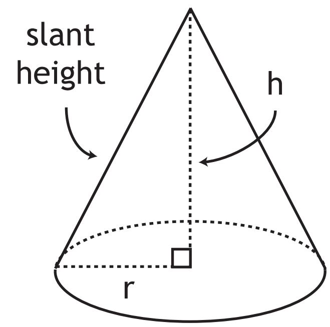


b) What is Greg's profit if he sells 52 games?

c) How many games must Greg sell to break even?

Example 9

The surface area and volume of a right cone are:



$$SA = \pi r^2 + \pi rs$$
$$V = \frac{1}{3}\pi r^2 h$$

where r is the radius of the circular base, h is the height of the apex, and s is the slant height of the side of the cone.

A particular cone has a height that is $\sqrt{3}$ times larger than the radius.

a) Can we write the surface area and volume formulae as single-variable functions?

b) Express the apex height in terms of r.

c) Express the slant height in terms of r.

d) Rewrite both the surface area and volume formulae so they are single-variable functions of r.

e) Use a function operation to determine the surface area to volume ratio of the cone.

f) If the radius of the base of the cone is 6 m, find the exact value of the surface area to volume ratio.

Example 1

Given the functions $f(x) = x - 3$ and $g(x) = x^2$:

Function Composition
(tables of values and two function machines)

a) Complete the table of values for $(f \circ g)(x)$. *same as $f(g(x))$*

x	*g(x)*	*f(g(x))*
-3		
-2		
-1		
0		
1		
2		
3		

b) Complete the table of values for $(g \circ f)(x)$. *same as $g(f(x))$*

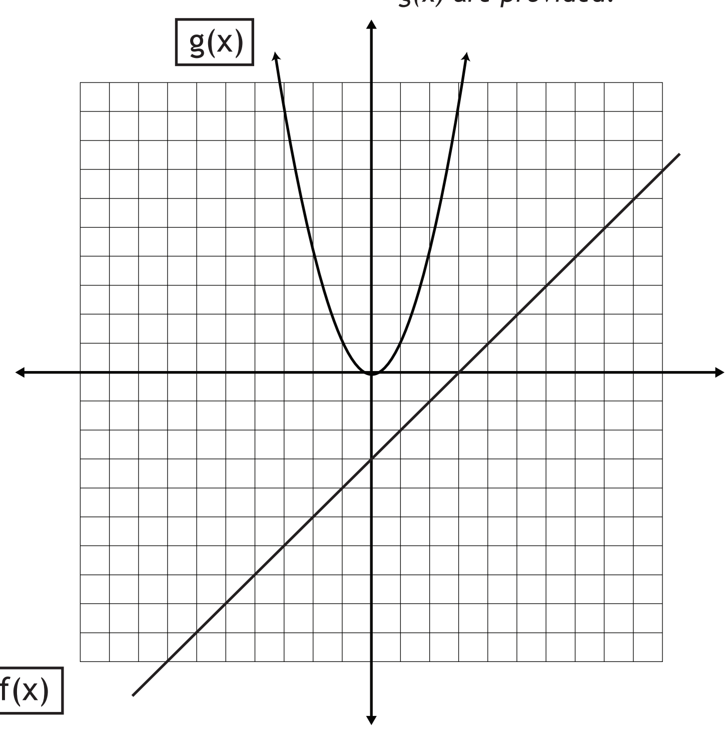
x	*f(x)*	*g(f(x))*
0		
1		
2		
3		

c) Does order matter when performing a composition?

d) Derive $m(x) = (f \circ g)(x)$.

e) Derive $n(x) = (g \circ f)(x)$.

f) Draw m(x) and n(x). *The graphs of f(x) and g(x) are provided.*



$$f \circ g = f(g(x))$$

Example 2

Given the functions $f(x) = x^2 - 3$ and $g(x) = 2x$, evaluate each of the following:

Function Composition
(numeric solution)

a) $m(3) = (f \circ g)(3)$

b) $n(1) = (g \circ f)(1)$

c) $p(2) = (f \circ f)(2)$

d) $q(-4) = (g \circ g)(-4)$

Example 3 Given the functions $f(x) = x^2 - 3$ and $g(x) = 2x$ *(these are the same functions found in Example 2)*, find each composite function.

Function Composition
(algebraic solution)

a) $m(x) = (f \circ g)(x)$

b) $n(x) = (g \circ f)(x)$

c) $p(x) = (f \circ f)(x)$

d) $q(x) = (g \circ g)(x)$

e) Using the composite functions derived in parts (a - d), evaluate m(3), n(1), p(2), and q(-4).
Do the results match the answers in Example 2?

$$f \circ g = f(g(x))$$

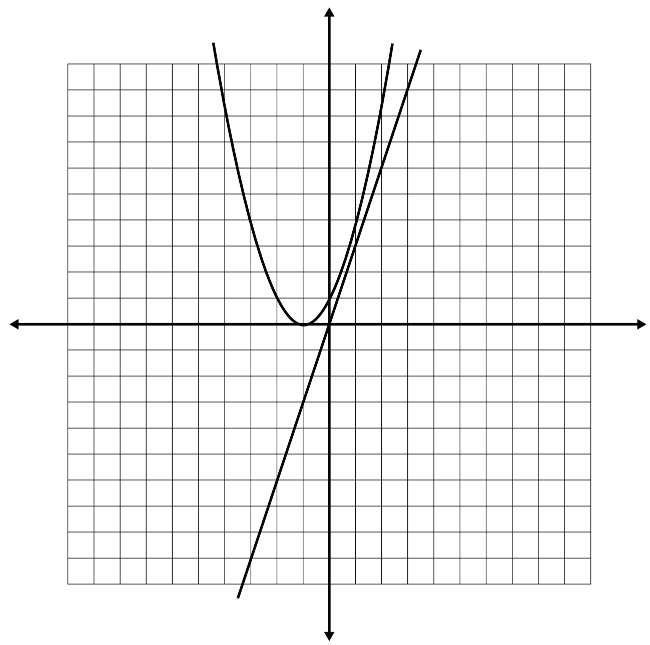
Example 4 Given the functions f(x) and g(x), find each composite function. Make note of any transformations as you complete your work.

Function Composition and Transformations

$f(x) = (x + 1)^2$ $g(x) = 3x$

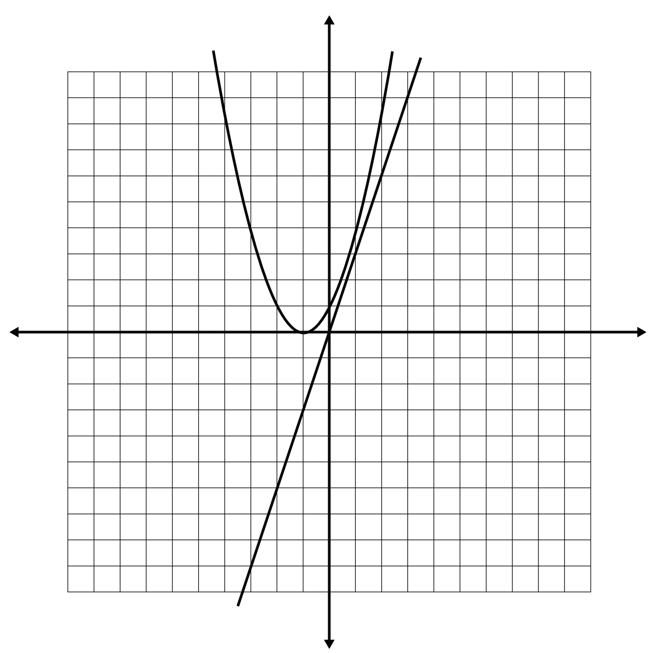
a) $m(x) = (f \circ g)(x)$

Transfomation:



b) $n(x) = (g \circ f)(x)$

Transfomation:



$$f \circ g = f(g(x))$$

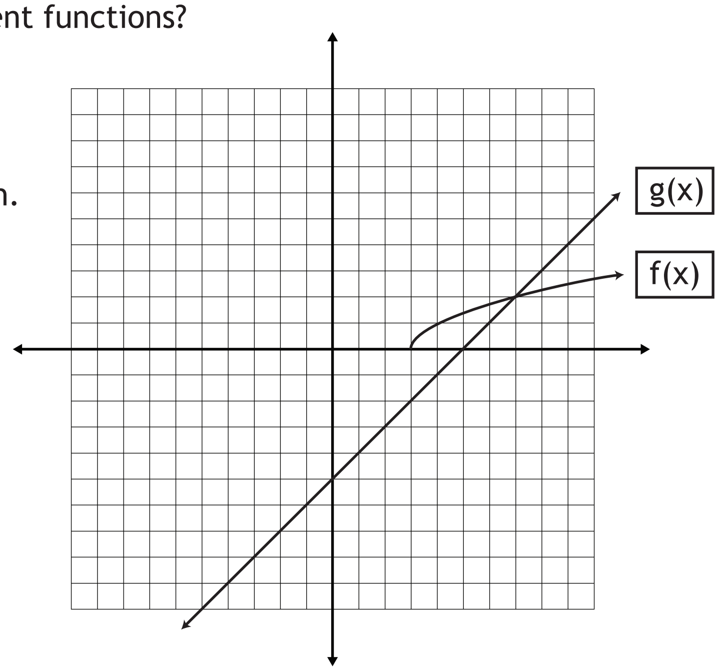
Transformations and Operations
LESSON FIVE - *Function Composition*
Lesson Notes

Example 5

Given the functions f(x) and g(x), find the composite function m(x) = (f ◦ g)(x) and draw the graph. How does the domain of the composite function compare to the domain of the component functions?

Domain of Composite Functions

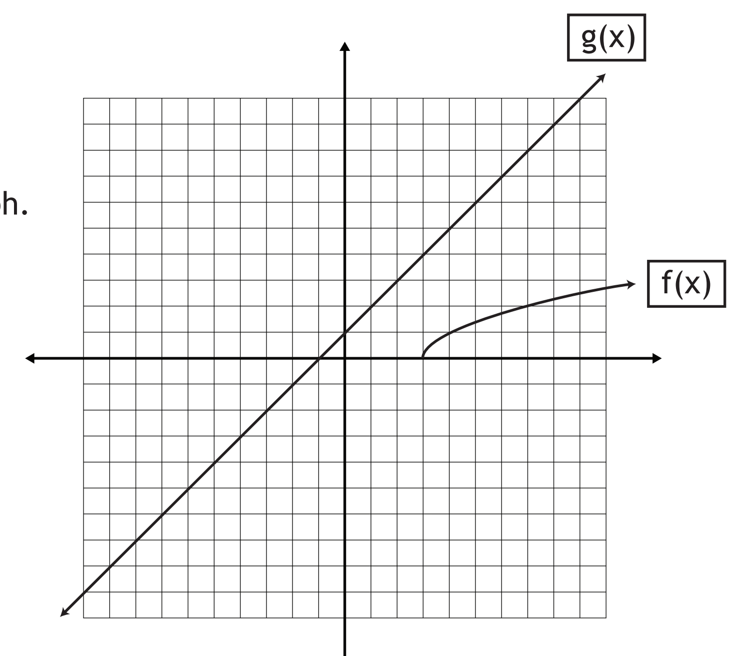
a) $f(x) = \sqrt{x - 3}$
$g(x) = x - 5$



i) Derive m(x) = (f ◦ g)(x) and draw the graph.

ii) State the domain of m(x).

b) $f(x) = \sqrt{x - 3}$
$g(x) = x + 1$



i) Derive m(x) = (f ◦ g)(x) and draw the graph.

ii) State the domain of m(x).

Transformations and Operations
LESSON FIVE - *Function Composition*
Lesson Notes

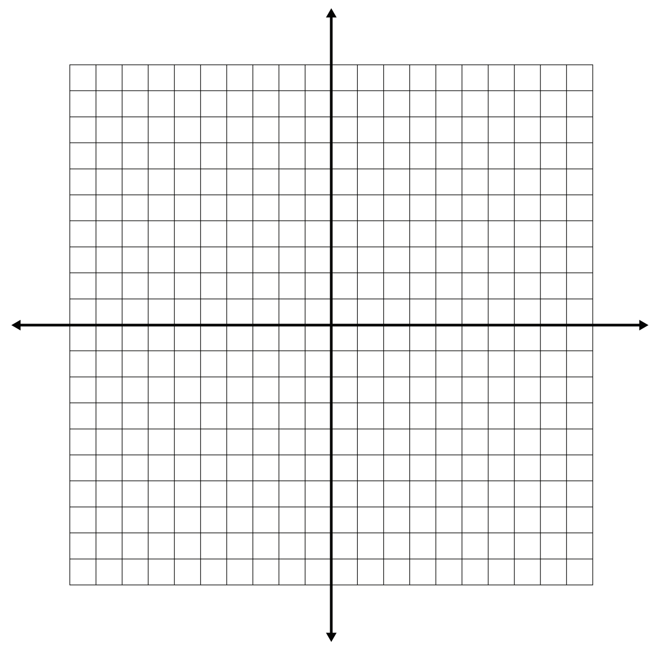
$f \circ g = f(g(x))$

Example 6 Given the functions f(x), g(x), m(x), and n(x), find each composite function. State the domain of the composite function and draw its graph.

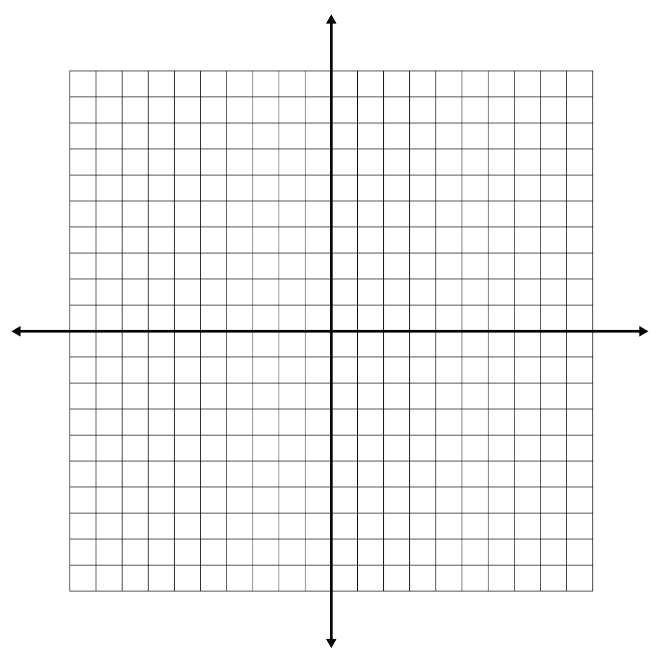
Function Composition
(three functions)

$f(x) = \sqrt{x}$ $\quad g(x) = \dfrac{1}{x}$ $\quad m(x) = |x|$ $\quad n(x) = x + 2$

a) $h(x) = [g \circ m \circ n](x)$



b) $h(x) = [n \circ f \circ n](x)$



$f \circ g = f(g(x))$

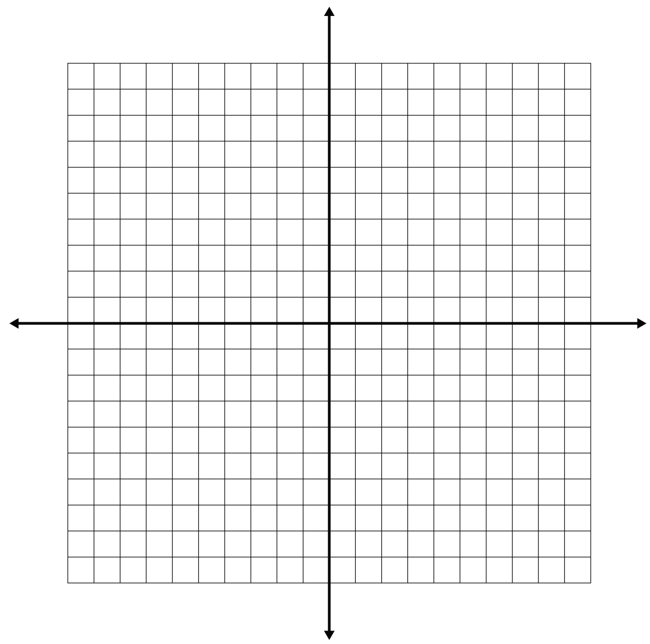
Transformations and Operations
LESSON FIVE - *Function Composition*
Lesson Notes

Example 7 Given the functions f(x), g(x), m(x), and n(x), find each composite function. State the domain of the composite function and draw its graph.

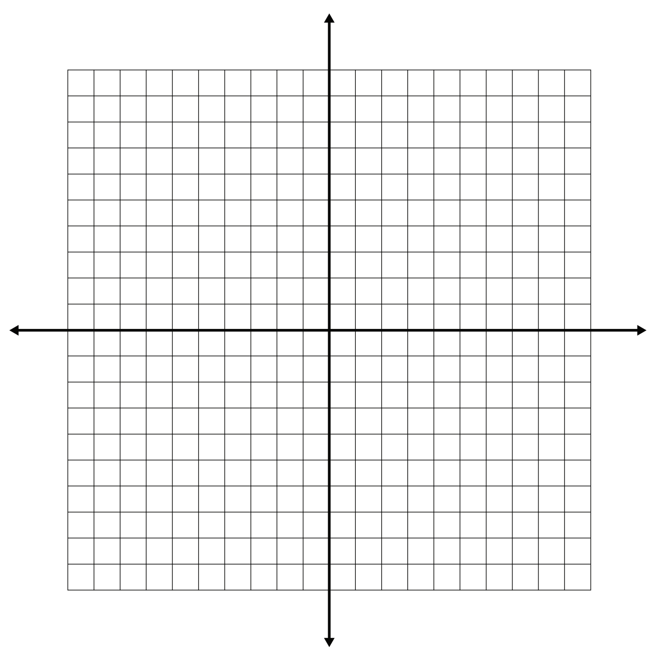
Function Composition
(with additional operations)

$f(x) = \sqrt{x}$ $\quad g(x) = \dfrac{1}{x}$ $\quad m(x) = |x|$ $\quad n(x) = x + 2$

a) $h(x) = [(gg) \circ n](x)$



b) $h(x) = [f \circ (n + n)](x)$



$$f \circ g = f(g(x))$$

Example 8

Given the composite function $h(x) = (f \circ g)(x)$, find the component functions, $f(x)$ and $g(x)$.
(More than one answer is possible)

Components of a Composite Function

a) $h(x) = 2x + 2$

b) $h(x) = \dfrac{1}{x^2 - 1}$

c) $h(x) = (x + 1)^2 - 5(x + 1) + 1$

d) $h(x) = x^2 + 4x + 4$

e) $h(x) = 2\sqrt{\dfrac{1}{x}}$

f) $h(x) = |x|$

Example 9

Two functions are inverses if $(f^{-1} \circ f)(x) = x$. Determine if each pair of functions are inverses of each other.

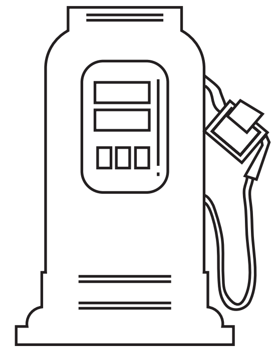
Composite Functions and Inverses

a) $f(x) = 3x - 2$ and $f^{-1}(x) = \frac{1}{3}x + \frac{2}{3}$

b) $f(x) = x - 1$ and $f^{-1}(x) = 1 - x$

$$f \circ g = f(g(x))$$

Example 10



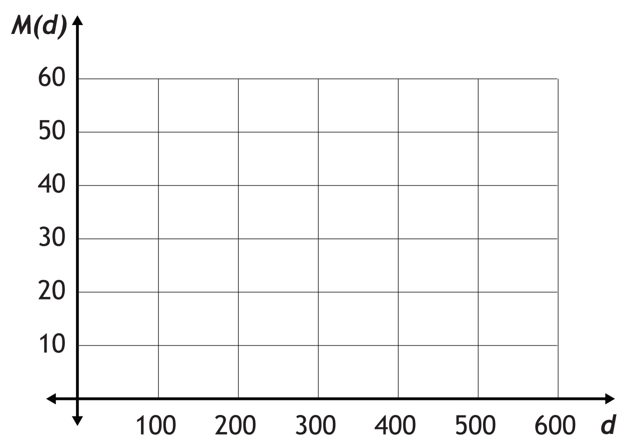
The price of 1 L of gasoline is $1.05. On a level road, Darlene's car uses 0.08 L of fuel for every kilometre driven.

a) If Darlene drives 50 km, how much did the gas cost to fuel the trip? How many steps does it take to solve this problem *(without composition)*?

b) Write a function, V(d), for the volume of gas consumed as a function of the distance driven.

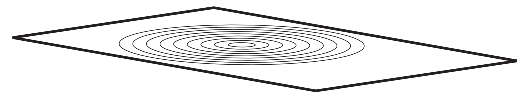
c) Write a function, M(V), for the cost of the trip as a function of gas volume.

d) Using function composition, combine the functions from parts b & c into a single function, M(d), where M is the money required for the trip. Draw the graph.



e) Solve the problem from part (a) again, but this time use the function derived in part (d). How many steps does the calculation take now?

Example 11



A pebble dropped in a lake creates a circular wave that travels outward at a speed of 30 cm/s.

a) Use function composition to derive a function, A(t), that expresses the area of the circular wave as a function of time.

b) What is the area of the circular wave after 3 seconds?

c) How long does it take for the area enclosed by the circular wave to be 44100π cm^2?
What is the radius of the wave?

Example 12

$CAD

$USD

€

¥

£

The exchange rates of several currencies on a particular day are listed below:

American Dollars = 1.03 × Canadian Dollars

Euros = 0.77 × American Dollars

Japanese Yen = 101.36 × Euros

British Pounds = 0.0083 × Japanese Yen

a) Write a function, a(c), that converts Canadian dollars to American dollars.

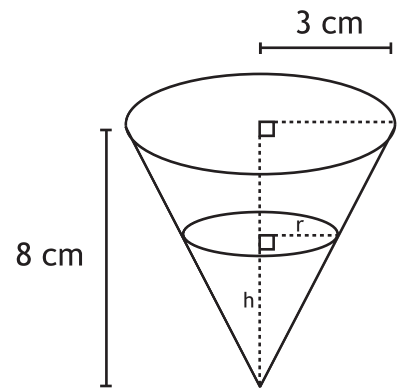
b) Write a function, j(a), that converts American Dollars to Japanese Yen.

c) Write a function, b(a), that converts American Dollars to British Pounds.

d) Write a function, b(c), that converts Canadian Dollars to British Pounds.

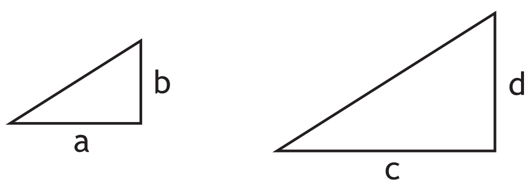
Example 13

A drinking cup from a water fountain has the shape of an inverted cone. The cup has a height of 8 cm, and a radius of 3 cm. The water in the cup also has the shape of an inverted cone, with a radius of r and a height of h.



The diagram of the drinking cup shows two right triangles: a large triangle for the entire height of the cup, and a smaller triangle for the water in the cup. The two triangles have identical angles, so they can be classified as similar triangles.

Reminder: In similar triangles, the ratios of corresponding sides are equal.



$$\frac{d}{b} = \frac{c}{a}$$

Reminder: The volume of a cone is:

$$V = \frac{1}{3}\pi r^2 h$$

a) Use similar triangle ratios to express r as a function of h.

b) Derive the composite function, $V_{water}(h) = (V_{cone} \circ r)(h)$, for the volume of the water in the cone.

c) If the volume of water in the cone is 3π cm^3, determine the height of the water.

$$f \circ g = f(g(x))$$

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$y = b^x$

Exponential and Logarithmic Functions
LESSON ONE - *Exponential Functions*
Lesson Notes

Example 1

Exponential Functions

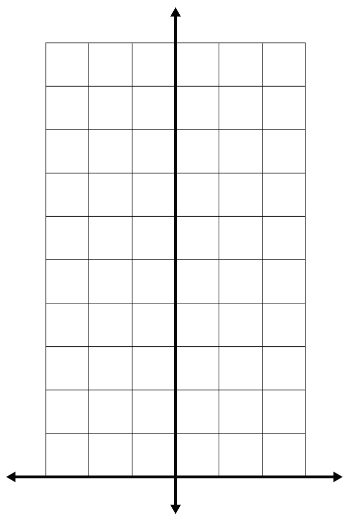
Graphing Exponential Functions

For each exponential function:

i) Complete the table of values and draw the graph.
ii) State the domain, range, intercepts, and the equation of the asymptote.

a) $y = 2^x$

x	y
-3	
-2	
-1	
0	
1	
2	
3	



Domain:

Range:

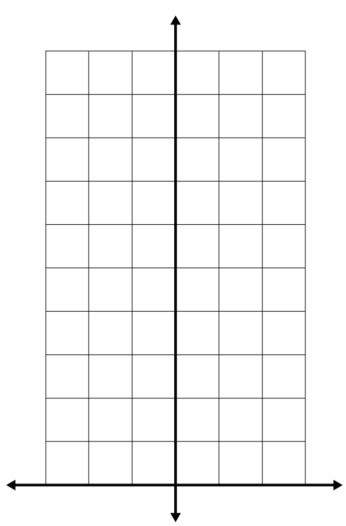
x-intercept:

y-intercept:

Asymptote:

b) $y = 3^x$

x	y
-3	
-2	
-1	
0	
1	
2	
3	



Domain:

Range:

x-intercept:

y-intercept:

Asymptote:

Set-Builder Notation

A ***set*** is simply a collection of numbers, such as {1, 4, 5}. We use ***set-builder notation*** to outline the rules governing members of a set.

{x | x ε R, x ≥ -1}

State the variable. List conditions on the variable.

In words: "The variable is x, such that x can be any real number with the condition that x ≥ -1".

As a shortcut, set-builder notation can be reduced to just the most important condition.

x ≥ -1

While this resource uses the shortcut for brevity, as set-builder notation is covered in previous courses, Math 30-1 students *are* expected to know how to read and write full set-builder notation.

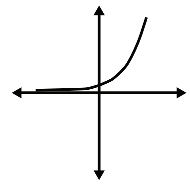
Interval Notation

Math 30-1 students are expected to know that domain and range can be expressed using ***interval notation***.

() - Round Brackets: Exclude point from interval.
[] - Square Brackets: Include point in interval.
Infinity ∞ always gets a round bracket.

Examples: x ≥ -5 becomes [-5, ∞);
1 < x ≤ 4 becomes (1, 4];
x ε R becomes (-∞ , ∞);
-8 ≤ x < 2 or 5 ≤ x < 11
becomes [-8, 2) U [5, 11),
where U means "or", or ***union of sets***;
x ε R, x ≠ 2 becomes (-∞ , 2) U (2, ∞);
-1 ≤ x ≤ 3, x ≠ 0 becomes [-1 , 0) U (0, 3].

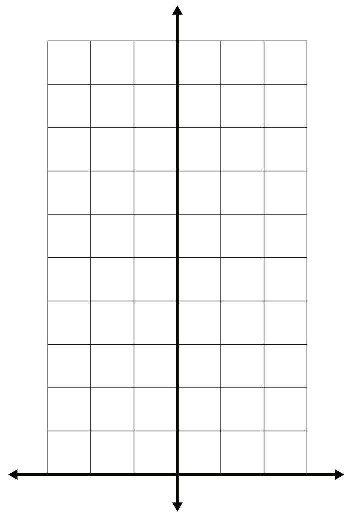
Lesson Notes



$y = b^x$

c) $y = \left(\frac{1}{2}\right)^x$

x	y
-3	
-2	
-1	
0	
1	
2	
3	



Domain:

Range:

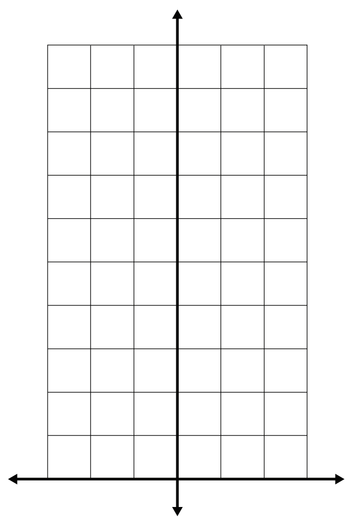
x-intercept:

y-intercept:

Asymptote:

d) $y = \left(\frac{1}{3}\right)^x$

x	y
-3	
-2	
-1	
0	
1	
2	
3	



Domain:

Range:

x-intercept:

y-intercept:

Asymptote:

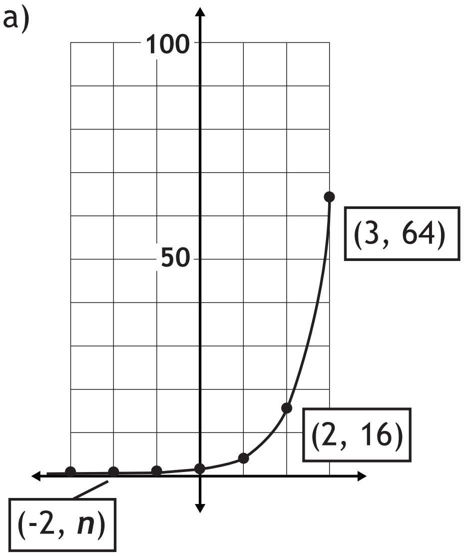
e) Define *exponential function*. Are the functions $y = 0^x$ and $y = 1^x$ considered exponential functions? What about $y = (-1)^x$?

Example 2

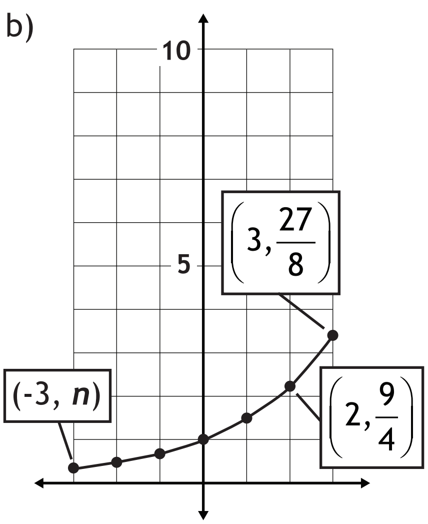
Determine the exponential function corresponding to each graph, then use the function to find the unknown.
All graphs in this example have the form $y = b^x$.

Exponential Function of a Graph. *($y = b^x$)*

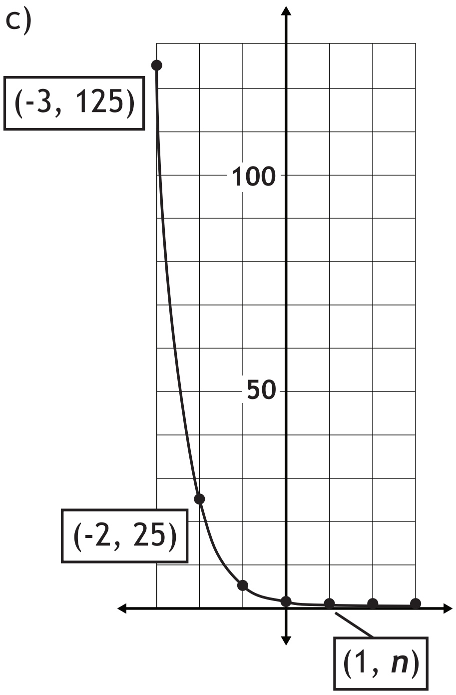
a)



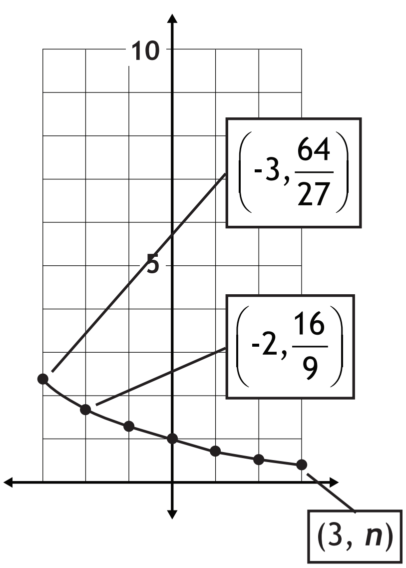
b)



c)


(-3, 125)
100
50
(-2, 25)
(1, *n*)

d)


10
5
$\left(-3, \frac{64}{27}\right)$
$\left(-2, \frac{16}{9}\right)$
(3, *n*)

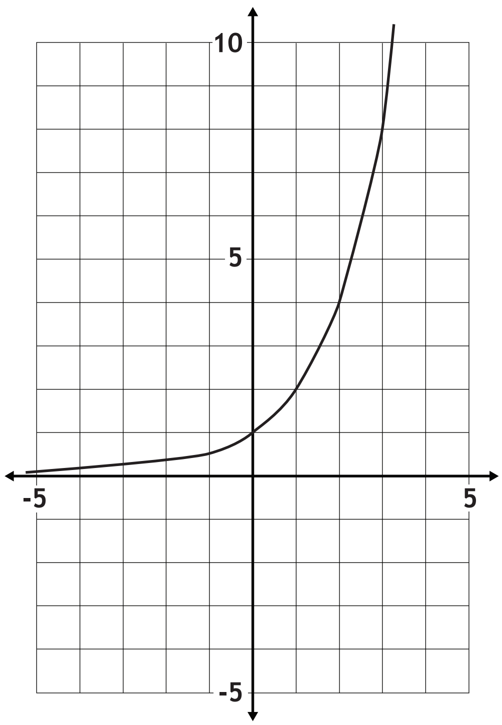
$y = b^x$

Example 3

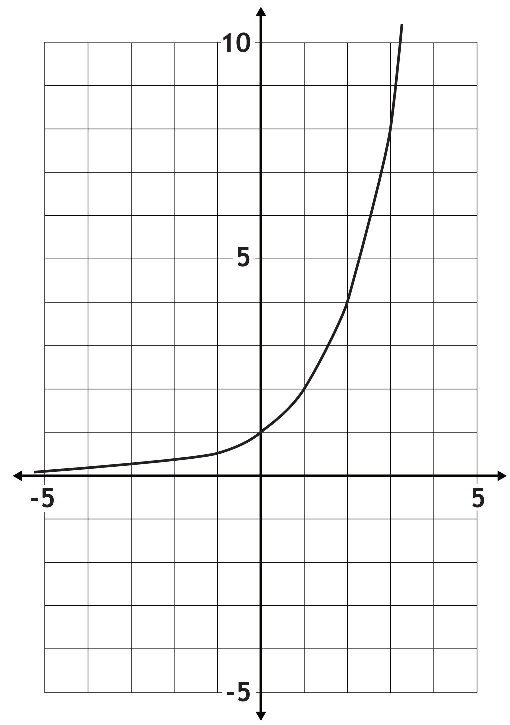
Draw the graph. The graph of $y = 2^x$ is provided as a convenience. State the domain, range, and equation of the asymptote.

Transformations

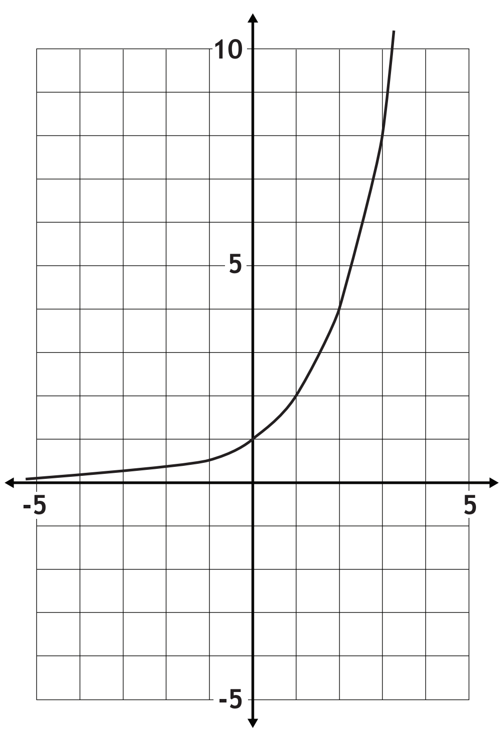
a) $y = 3(2)^x$



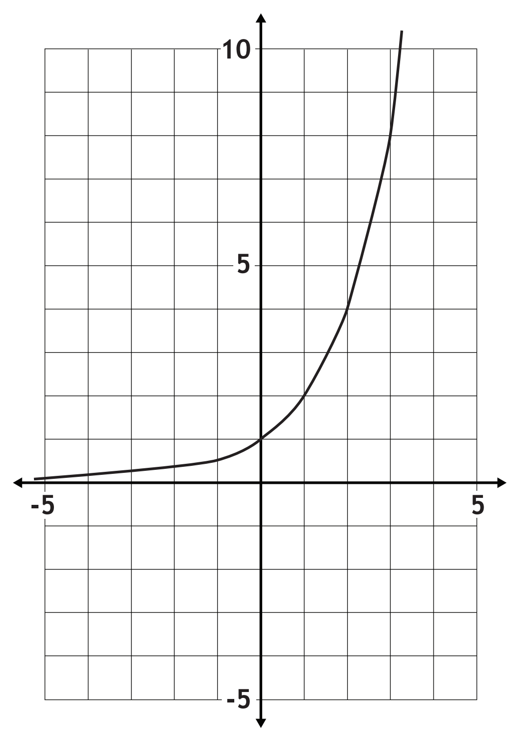
b) $y = 2^{\frac{x}{4}}$



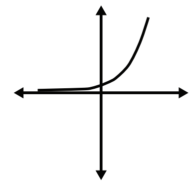
c) $y = 2^x + 3$



d) $y = 2^{x-1}$



Exponential and Logarithmic Functions
LESSON ONE- *Exponential Functions*
Lesson Notes



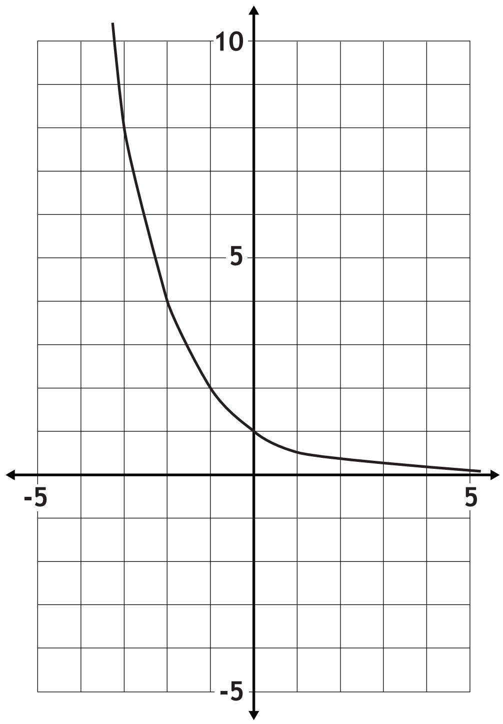
Example 4

Draw the graph. The graph of $y = (1/2)^x$ is provided as a convenience. State the domain, range, and equation of the asymptote.

Transformations

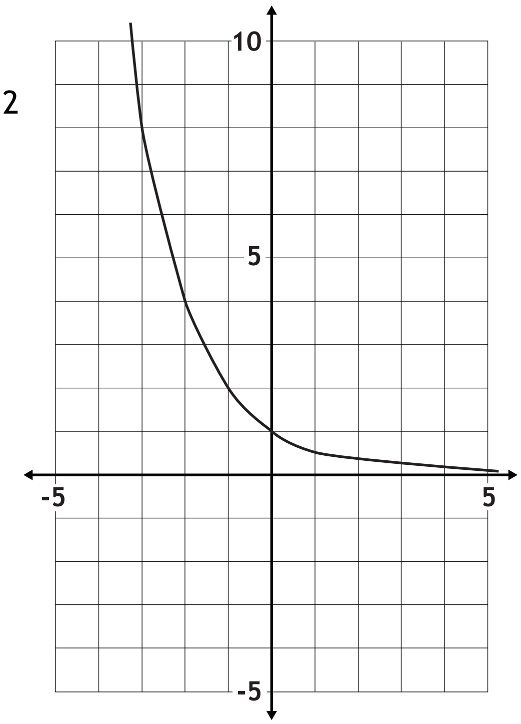
a)

$$y = 2\left(\frac{1}{2}\right)^x - 4$$



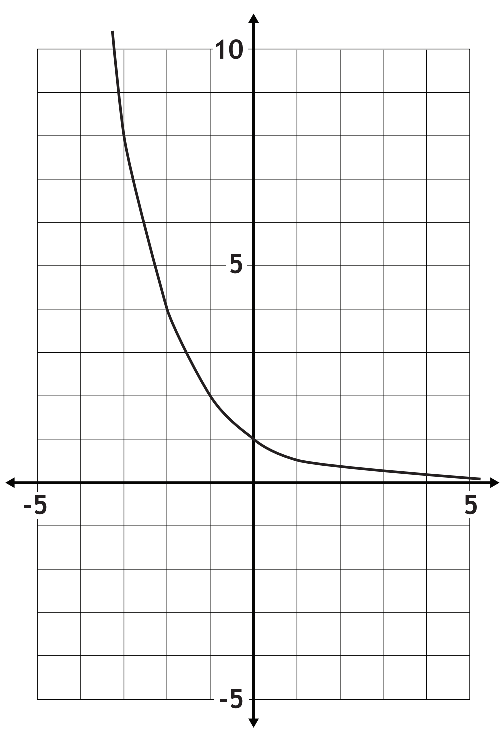
b)

$$y = \left(\frac{1}{2}\right)^{x+3} - 2$$



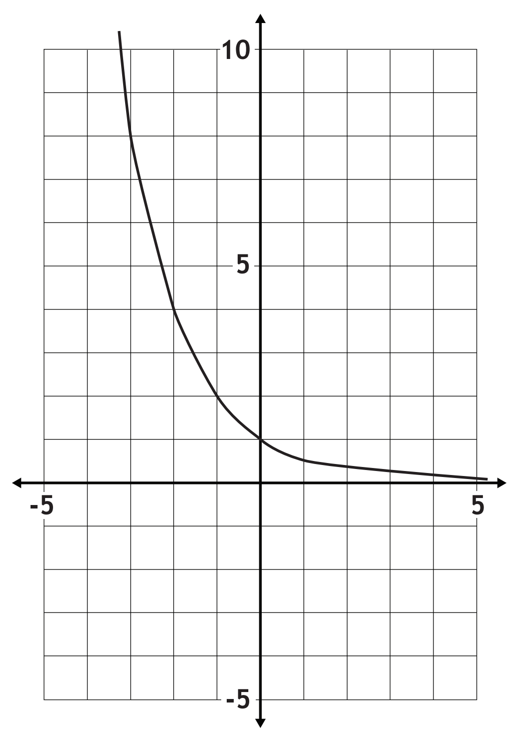
c)

$$y = \left(\frac{1}{2}\right)^{\frac{1}{2}(x-1)}$$



d)

$$y = \left(\frac{1}{2}\right)^{2x+6}$$

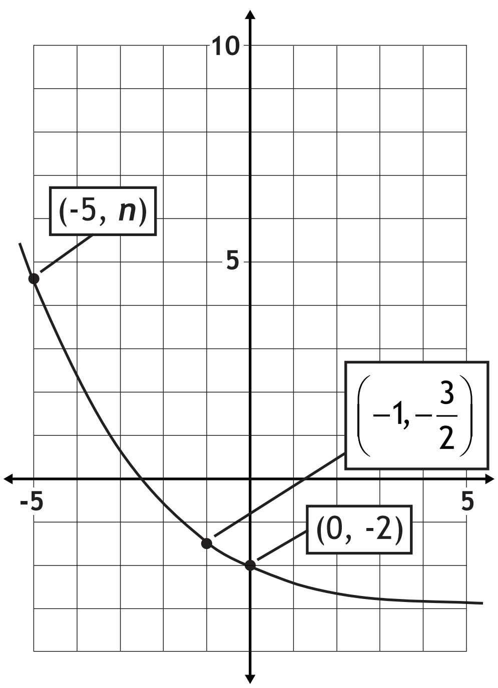


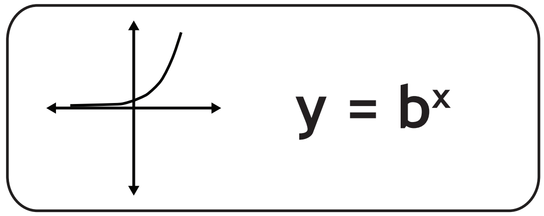
Example 5

Determine the exponential function corresponding to each graph, then use the function to find the unknown. Both graphs in this example have the form $y = ab^x + k$.

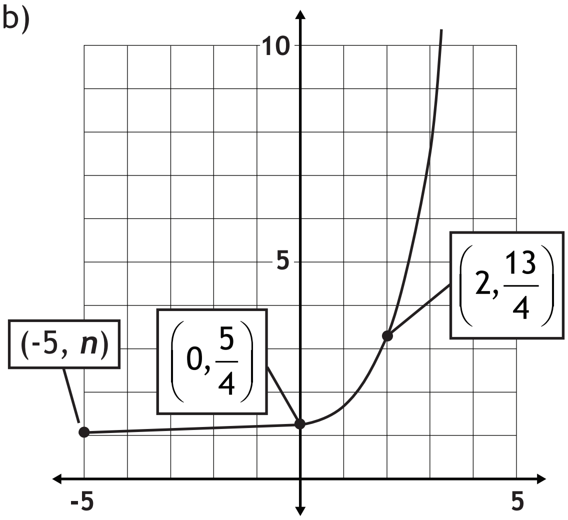
Exponential Function of a Graph. *($y = ab^x + k$)*

a)





b)



Example 6 Answer each of the following questions. Assorted Questions

a) What is the y-intercept of $f(x) = ab^{x-4}$?

b) The point $\left(-1, \frac{5}{3}\right)$ exists on the graph of $y = a(5)^x$. What is the value of a?

c) If the graph of $y = \left(\frac{1}{3}\right)^x$ is stretched vertically so it passes through the point $\left(2, \frac{1}{12}\right)$, what is the equation of the transformed graph?

d) If the graph of $y = 2^x$ is vertically translated so it passes through the point (3, 5), what is the equation of the transformed graph?

e) If the graph of $y = 3^x$ is vertically stretched by a scale factor of 9, can this be written as a horizontal translation?

f) Show algebraically that each pair of graphs are identical.

i) $y = 25(5)^x$ and $y = 5^{x+2}$

ii) $y = \frac{1}{8}(2)^x$ and $y = 2^{x-3}$

iii) $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$

iv) $y = \frac{64}{27}\left(\frac{3}{4}\right)^{-x}$ and $y = \left(\frac{4}{3}\right)^{x+3}$

v) $y = \frac{3}{4}\left(\frac{1}{3}\right)^x$ and $y = \frac{1}{4}\left(\frac{1}{3}\right)^{x-1}$

Example 7 Solving equations where x is in the base.

Raising Reciprocals

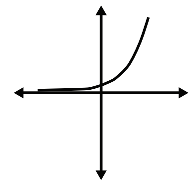
a) $x^3 = 8$

b) $x^{\frac{1}{4}} = 2$

c) $x^{-\frac{3}{5}} = 27$

d) $(16x)^{\frac{2}{3}} = 4$

Lesson Notes



$y = b^x$

Example 8 Solving equations where x is in the exponent.

Common Base

a) $2^{2x+1} = 8^{x-1}$

b) $2^{3x} = 32^{x-2}$

c) $8^{x-1} = 16^{x-2}$

d) $9^{\frac{x}{2}} = 27^{x-4}$

e) Determine x and y: $8^x = \frac{1}{64}$, $25^{x+y} = 125$

f) Determine m and n: $27^{2m-n} = \frac{1}{9}$, $49^{3m-2n} = 7$

Example 9 Solving equations where *x* is in the exponent.

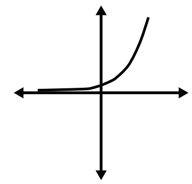
Common Base
(fractional base)

a) $\left(\frac{1}{6}\right)^{x} = 36$

b) $\left(\frac{125}{8}\right)^{x-2} = \left(\frac{25}{4}\right)^{2x-5}$

c) $\left(\frac{9}{4}\right)^{x-4} = \left(\frac{8}{27}\right)^{2x}$

d) $\left(\frac{16}{81}\right)^{6x} = \left(\frac{27}{8}\right)^{-10x+1}$



Example 10

Solving equations where *x* is in the exponent.

Common Base
(fractional exponents)

a) $3^{\frac{2x}{3}} = 9^{x-4}$

b) $25^{\frac{10+x}{3}-2} = 125^{\frac{2x}{5}}$

c) $\left(\frac{1}{8}\right)^{\frac{x}{9}-6} = 4^{\frac{x}{2}-3}$

d) $\left(\frac{3}{4}\right)^{\frac{2}{3}(x+3)} = \left(\frac{64}{27}\right)^{\frac{x}{3}-9}$

Example 11 Solving equations where *x* is in the exponent.

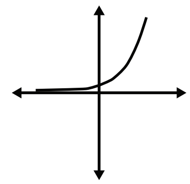
Common Base
(multiple powers)

a) $16^{3x} = \left(2^{5x+2}\right)\left(8^{2x}\right)$

b) $27^{x+1} = \left(3^{x-3}\right)\left(9^{x+3}\right)$

c) $125\left(\frac{4}{5}\right)^{2x+1} = 64$

d) $8^{x+1} = \frac{1}{64^{1-x}}$



$y = b^x$

Example 12

Solving equations where x is in the exponent.

Common Base
(radicals)

a) $3^x = 9\sqrt{3}$

b) $5^x = 125\sqrt{5}$

c) $64^{x-2} = \left(\sqrt[4]{4}\right)^{3x+3}$

d) $3^{4x} = \left(\sqrt[3]{9}\right)^{2x+4}$

Example 13 Solving equations where *x* is in the exponent. Factoring

a) $4^{2x} - 6(4)^x + 8 = 0$

b) $2(2)^{-2x} - 9(2)^{-x} + 4 = 0$

c) $2^{x+3} + 2^{x+4} = 96$

d) $3^x - 3^{x-1} = 162$

Example 14

Solving equations where x is in the exponent.

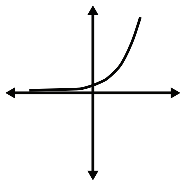
No Common Base
(use technology)

a) $3^x = 7$

b) $\left(\frac{1}{2}\right)^x = -3$

c) $2(4)^{x-1} = 6$

d) $12\left(\frac{1}{2}\right)^{x-1} = 3$

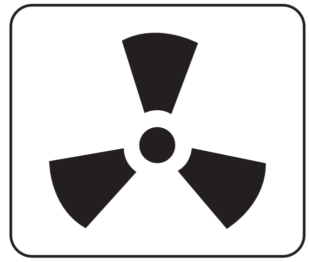


$y = b^x$

Example 15 A 90 mg sample of a radioactive isotope has a half-life of 5 years.

$y = ab^{\frac{t}{P}}$

a) Write a function, *m(t)*, that relates the mass of the sample, *m*, to the elapsed time, *t*.



b) What will be the mass of the sample in 6 months?

Logarithmic Solutions

Some of these examples provide an excellent opportunity to use logarithms.

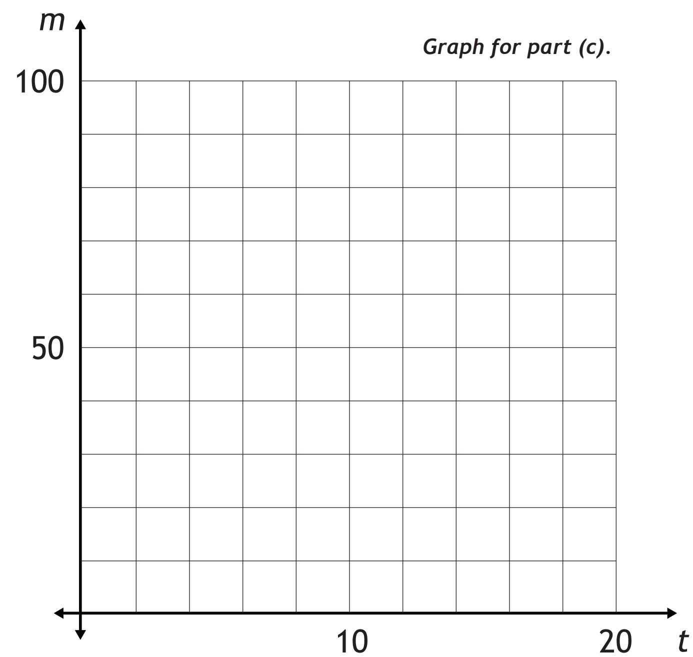
Logarithms are not a part of this lesson, but it is recommended that you return to these examples at the end of the unit and complete the logarithm portions.

c) Draw the graph for the first 20 years.

d) How long will it take for the sample to have a mass of 0.1 mg?

Solve Graphically	Solve with Logarithms

Graph for part (c).

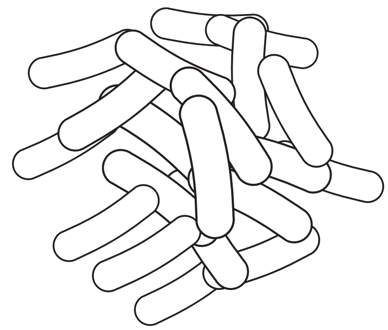


Example 16

A bacterial culture contains 800 bacteria initially and doubles every 90 minutes.

$y = ab^{\frac{t}{P}}$

a) Write a function, *B(t)*, that relates the quantity of bacteria, *B*, to the elapsed time, *t*.



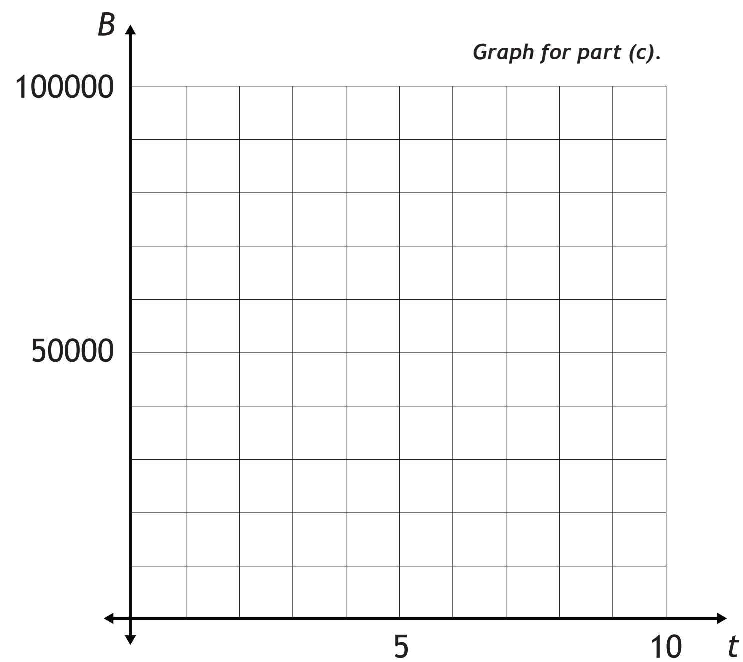
b) How many bacteria will exist in the culture after 8 hours?

c) Draw the graph for the first ten hours.

d) How long ago did the culture have 50 bacteria?

Solve Graphically

Solve with Logarithms

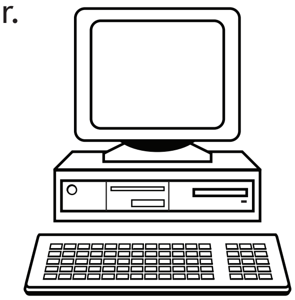


Graph for part (c).

Example 17

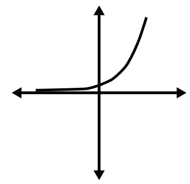
In 1990, a personal computer had a processor speed of 16 MHz. In 1999, a personal computer had a processor speed of 600 MHz. Based on these values, the speed of a processor increased at an average rate of 44% per year.

$$y = ab^{\frac{t}{P}}$$



a) Estimate the processor speed of a computer in 1994 (t = 4).
How does this compare with actual processor speeds (66 MHz) that year?

b) A computer that cost $2500 in 1990 depreciated at a rate of 30% per year.
How much was the computer worth four years after it was purchased?



$y = b^x$

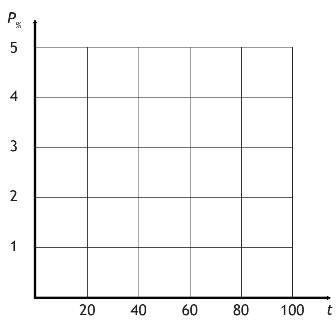
Example 18 A city with a population of 800,000 is projected to grow at an annual rate of 1.3%.

$y = ab^{\frac{t}{P}}$

a) Estimate the population of the city in 5 years.

b) How many years will it take for the population to double?

Solve Graphically

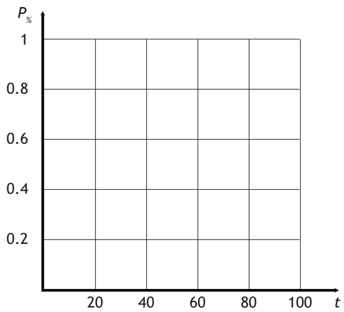


Solve with Logarithms

c) If projections are incorrect, and the city's population *decreases* at an annual rate of 0.9%, estimate how many people will leave the city in 3 years.

d) How many years will it take for the population to be reduced by half?

Solve Graphically



Solve with Logarithms

Example 19 \$500 is placed in a savings account that compounds interest annually at a rate of 2.5%.

$$y = ab^{\frac{t}{P}}$$

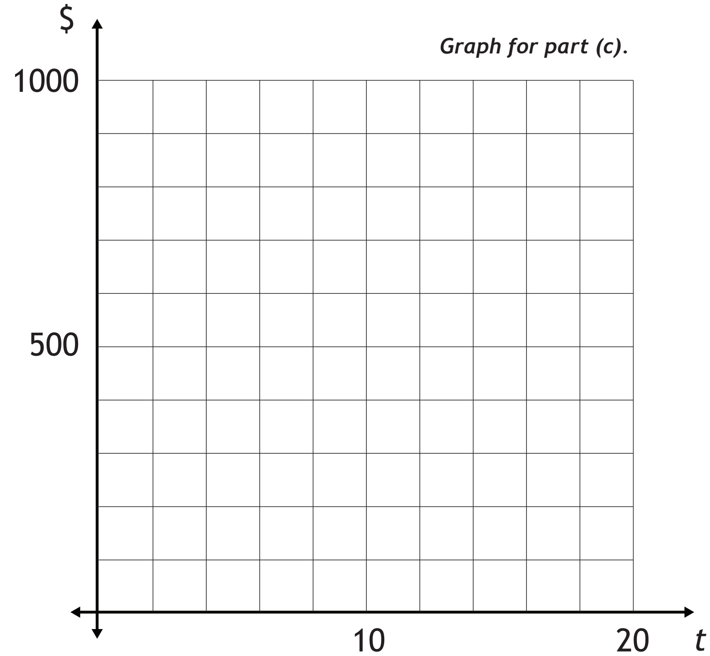
a) Write a function, $A(t)$, that relates the amount of the investment, A, with the elapsed time t.

b) How much will the investment be worth in 5 years? How much interest has been received?

c) Draw the graph for the first 20 years.

d) How long does it take for the investment to double?

Solve Graphically | **Solve with Logarithms**



Graph for part (c).

e) Calculate the amount of the investment in 5 years if compounding occurs i) semi-annually, ii) monthly, and iii) daily. Summarize your results in the table.

Future amount of $500 invested for 5 years and compounded:

Annually	*Use answer from part b.*
Semi-Annually	
Monthly	
Daily	

Lesson Notes

Example 1

Introduction to Logarithms.

Logarithm Components

a) Label the components of $\log_B A = E$ and $A = B^E$.

$$\log_B A = E$$

____
____
____

$$A = B^E$$

____
____
____

b) Evaluate each logarithm.

i) $\log_2 1 = \square$

$\log_2 2 = \square$

$\log_2 4 = \square$

$\log_2 8 = \square$

ii) $\log 1 = \square$

$\log 10 = \square$

$\log 100 = \square$

$\log 1000 = \square$

c) Which logarithm is bigger?

i) $\log_2 1$ or $\log_4 2$

ii) $\log_3\left(\frac{1}{9}\right)$ or $\log_9\left(\frac{1}{3}\right)$

$\log_B A = E$

Example 2 Order each set of logarithms from least to greatest.

Ordering Logarithms

a) $\log 10, \quad \log_2 16, \quad \log_9\left(\frac{1}{3}\right), \quad \log_{16}\left(\frac{1}{2}\right), \quad \log_5 1$

b) $\log_{\frac{1}{3}} 27, \quad \log_{\frac{1}{4}} 8, \quad \log_{\frac{1}{8}}\left(\frac{1}{2}\right), \quad \log_{\frac{1}{4}}\left(\frac{1}{2}\right), \quad \log_{\frac{1}{8}}\left(\frac{1}{8}\right)$

c) $\log_3 25, \quad \log_6 7, \quad \log_{\frac{1}{4}}\left(\frac{1}{15}\right), \quad \log_8 3$ (Estimate the order using benchmarks)

Example 3

Convert each equation from logarithmic to exponential form.

Express answers so y is isolated on the left side.

Logarithmic to Exponential Form
(The Seven Rule)

$\log_b y = x \rightarrow b^x = y$

a) $\log_2 y = x$

b) $2 = \log_4 y$

c) $a \log y = x$

d) $\log_3 (2y) = x$

e) $\frac{1}{2} = \log_x y$

f) $\log_2 (y - x) = 3$

g) $2 = \log_{x+1} (y + 1)$

h) $\log_3 (3y) = 2x$

Exponential and Logarithmic Functions
LESSON TWO - *Laws of Logarithms*
Lesson Notes

$\log_B A = E$

Example 4

Convert each equation from exponential to logarithmic form.

Express answers with the logarithm on the left side.

Exponential to Logarithmic Form
(A Base is Always a Base)

$\boldsymbol{b}^x = y \longrightarrow \log_{\boldsymbol{b}} \boxed{y} = \boxed{x}$

a) $y = x^2$

b) $10x^4 = y$

c) $y = \left(\frac{1}{3}\right)^x$

d) $\sqrt{x} = 3y$

e) $y = \sqrt[3]{\frac{x}{2}}$

f) $y = (x-3)^2$

g) $y = \frac{k^x}{k}$

h) $10^{y-x} = a$

Example 5 Evaluate each logarithm using change of base.

Evaluating Logarithms
(Change of Base)

a) $\log_4 64$

b) $\log_{\frac{2}{3}} \frac{8}{27}$

General Rule

$$\log_b a = \frac{\log_c a}{\log_c b}$$

For Calculator
(base-10)

$$\log_b a = \frac{\log a}{\log b}$$

c) $\log_{\sqrt{2}} 2$

d) $\log 100$

In parts (e - h), condense each expression to a single logarithm.

e) $\dfrac{\log 5}{\log 25}$

f) $\dfrac{\log \sqrt{3}}{\log 3}$

g) $\dfrac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{1}{3}\right)}$

h) $\left(\log_a x\right)\left(\log_x b\right)$

Lesson Notes

$\log_B A = E$

Example 6

Expand each logarithm using the product law.

> Expanding Logarithms
> *(Product Law)*
> $\log_b (M \times N) = \log_b M + \log_b N$

a) $\log(xy)$

b) $\log(x + y)$

c) $\log(3(x + 1))$

d) $\log(10x)$

In parts (e - h), condense each expression to a single logarithm.

e) $\log 3 + \log 4$

f) $\log\frac{2}{3} + \log\frac{3}{4}$

g) $\log x^2 + \log x^3$

h) $\log(x + 1) + \log(x - 2)$

$\log_B A = E$

Example 7

Expand each logarithm using the quotient law.

Expanding Logarithms
(Quotient Law)

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

a) $\log\left(\frac{x}{y}\right)$

b) $\log(x - y)$

c) $\log\left(\frac{x+1}{100}\right)$

d) $\log_3\left(\frac{x}{3(x+1)}\right)$

In parts (e - h), condense each expression to a single logarithm.

e) $\log 12 - \log 4$

f) $\log\frac{1}{3} - \log 2$

g) $\log x^5 - \log x^2$

h) $\log 2 + \log x - \log(x + 3)$

Lesson Notes

$$\log_B A = E$$

Example 8 Expand each logarithm using the power law.

Expanding Logarithms
(Power Law)

$\log_b\left(M^n\right) = n\log_b M$

a) $\log x^2$

b) $\left(\log x\right)^2$

c) $\log x^3 + \log x^4$

d) $\log x^{a+1}$

In parts (e - h), condense each expression to a single logarithm.

e) $3\log x$

f) $2\log\left(x-1\right)$

g) $3\log\left(2x^2\right)$

h) $5\log x - 3\log x$

$\log_B A = E$

Example 9

Expand each logarithm using the appropriate logarithm rule.

Expanding Logarithms
(Other Rules)

$\log_b x$ has the domain x > 0
$\log_b 1 = 0$
$\log_b b = 1$
$b^{\log_b x} = x$
$\log_b b^x = x$

a) $\log_2 0$

b) $\log(-3)$

c) $\log_2 1$

d) $\log_4 4$

e) $5^{\log_5 x}$

f) $\log_2 2^x$

g) $\log_5 25^k$

h) $\log_a \left(\sqrt{a}\right)^k$

Exponential and Logarithmic Functions
LESSON TWO - *Laws of Logarithms*
Lesson Notes

$$\log_B A = E$$

Example 10 Use logarithm laws to answer each of the following questions.

Substitution Questions

a) If $10^k = 4$, then $10^{1 + 2k} =$

b) If $3^a = k$, then $\log_3 k^4 =$

c) If $\log_b 4 = k$, then $\log_b 16 =$

d) If $\log_2 a = h$, then $\log_4 a =$

e) If $\log_b h = 3$ and $\log_b k = 4$, then $\log_b\left(\frac{1}{hk}\right) =$

f) If $\log_h 4 = 2$ and $\log_8 k = 2$, then $\log_2(hk) =$

g) Write $\log x + 1$ as a single logarithm.

h) Write $3 + \log_2 x$ as a single logarithm.

Example 11 Solving Equations. Express answers using exact values.

Solving Exponential Equations
(No Common Base)

a) $3^x = 4$

b) $5^x = -2$

c) $2 \times 5^{x+2} = 7$

d) $\left(\frac{2}{5}\right)^{x-3} = \frac{1}{3}$

Lesson Notes

$$\log_B A = E$$

Example 12 Solving Equations. Express answers using exact values.

Solving Exponential Equations
(No Common Base)

a) $6^{5x} = 3^{2x-1}$

b) $2^{x+3} = 3^{2x-1}$

c) $\dfrac{4^{2x-1}}{3} = 5^x$

d) $2 \times 3^{x+3} = 6^{3x}$

Example 13 Solving Equations. Express answers using exact values.

Solving Logarithmic Equations
(One Solution)

a) $3\log x + 5 = 8$

b) $2\log_5 3 = \log_5(x+1)$

c) $\log_3(x-2) = \log_3(3x+2)$

d) $\log_3 x - \log_3 2 = \log_3 7$

$$\log_B A = E$$

Example 14 Solving Equations. Express answers using exact values.

Solving Logarithmic Equations
(Multiple Solutions)

a) $\log_2 x + \log_2 (x + 2) = 3$

b) $\log_2 (x - 1) + \log_2 (x - 2) - \log_2 3 = 2$

c) $\log x^2 + \log 3 = \log 2x$

d) $\log_4 (x^2 + 1) - \log_4 6 = \log_4 5$

Example 15 Solving Equations. Express answers using exact values.

Solving Logarithmic Equations
(Multiple Solutions)

a) $\log_{x-1} 25 = 2$

b) $2\log(x-3) = \log 4 + \log(6-x)$

c) $(\log x)^2 - 4\log x - 5 = 0$

d) $(\log x)^4 - 16 = 0$

$\log_B A = E$

Example 16 Assorted Questions. Express answers using exact values.

Assorted Mix I

a) Evaluate.

$\log_6 \sqrt[4]{6}$

b) Condense.

$\frac{1}{2}\log a - 3\log b - 2\log c$

c) Solve.

$3\log_2 x = 12$

d) Evaluate.

$\log_2\left(\log(10000)\right)$

e) Write as a logarithm.

$b^{\frac{5}{4}} = 2a$

f) Show that:

$\log_{\frac{1}{5}}\left(\frac{1}{x}\right) = \log_5 x$

g) If $\log_a 3 = x$ and $\log_a 4 = 12$,
then $\log_a 12^2 =$
(express answer in terms of x.)

h) Condense.

$2 + \frac{1}{3}\log_3 x$

Lesson Notes

$$\log_B A = E$$

Example 17 Assorted Questions. Express answers using exact values.

Assorted Mix II

a) Evaluate.

$\log_3 9 + \log_3 9^2 + \log_3 9^3$

b) Evaluate.

$\log_3 9 + (\log_3 9)^2 + (\log_3 9)^3$

c) What is one-third of 3^{234}?

d) Solve.

$8 = (x+1)^3$

e) Evaluate.

$\log_b \left(\frac{1}{b^{-100}} \right)$

f) Condense.

$\log_2 a + \log_4 b$

g) Solve.

$\log(x+2) + \log(x-1) = 1$

h) If xy = 8, then $5\log_2 x + 5\log_2 y =$

$\log_B A = E$

Example 18 Assorted Questions. Express answers using exact values.

Assorted Mix III

a) Evaluate.

$\log_2 \frac{1}{8}$

b) Solve.

$\log x - \log(x+5) = 1$

c) Condense.

$\log_4 8^x - \log_4 2^x$

d) Solve.

$(\log x)^2 = 2\log x$

e) Condense.

$$\left(\frac{1}{2}\right)^{\log_{\frac{1}{2}} a}\left(\frac{1}{2}\right)^{\log_{\frac{1}{2}} a}$$

f) Evaluate.

$$\log_9\left(\log_2 8\right)$$

g) Show that:

$$\log_{\frac{1}{2}} 81 = \log_2\left(\frac{1}{81}\right)$$

h) Condense.

$$\log_2\left(2x+1\right)+1$$

$$\log_B A = E$$

Example 19 Assorted Questions. Express answers using exact values.

Assorted Mix IV

a) Solve.

$\log_3(2x+1) - \log_3(x-1) = 1$

b) Condense.

$3(\log a + \log b)$

c) Solve.

$\log_{\sqrt{2}} x^4 + 4 = 12$

d) Condense.

$\log(a^2 + 2a + 1) - \log(a + 1)$

e) Evaluate.

$-\frac{1}{3}\log_2 64$

f) Solve.

$\log(2-x)+\log(2+x)=\log 3$

g) Evaluate.

$\frac{1}{4}\log_2 16+\log_3 \sqrt{27}$

h) Condense.

$3\log_{16} x+\frac{1}{2}$

$\log_B A = E$

Example 20 Assorted Questions. Express answers using exact values.

Assorted Mix V

a) Solve.

$\log(x+2) = \log x + \log 2$

b) Solve.

$2^{3x-1} = 5^{2x+3}$

c) Evaluate.

$\log_3 9^{99} + \log_4 64 + \log_a 1 + \log_{\frac{1}{2}} 8 + \log_{\sqrt{a}} \sqrt{a}$

d) Condense.

$\log x - 4\log\sqrt{x}$

e) Solve.

$$\log_4(\log_3 x) = \frac{1}{2}$$

f) Solve.

$$2\log x + 3\log x = 8$$

g) Condense.

$$4\log a - \frac{1}{2}\log b + \log c$$

h) Solve.

$$\log_{2x}(4x + 8) = 2$$

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Example 1

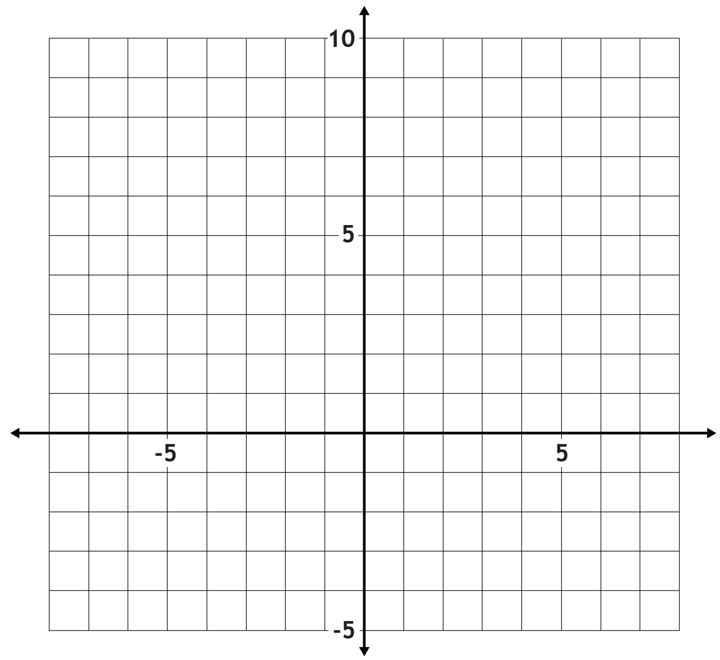
Logarithmic Functions

Graphing Logarithms

a) Draw the graph of $f(x) = 2^x$

b) Draw the inverse of $f(x)$.

c) Show algebraically that the inverse of $f(x) = 2^x$ is $f^{-1}(x) = \log_2 x$.



d) State the domain, range, intercepts, and asymptotes of both graphs.

	$y = 2^x$	$y = \log_2 x$
Domain		
Range		
x-intercept		
y-intercept		
Asymptote Equation		

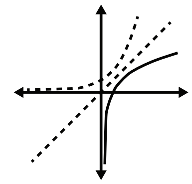
e) Use the graph to determine the value of:
i) $\log_2 0.5$, ii) $\log_2 1$, iii) $\log_2 2$, iv) $\log_2 7$

f) Are $y = \log_1 x$, $y = \log_0 x$, and $y = \log_{-2} x$ logarithmic functions?
What about $y = \log_{\frac{1}{10}} x$?

g) Define logarithmic function.

h) How can $y = \log_2 x$ be graphed in a calculator?

Exponential and Logarithmic Functions
LESSON THREE- *Logarithmic Functions*
Lesson Notes



$y = \log_b x$

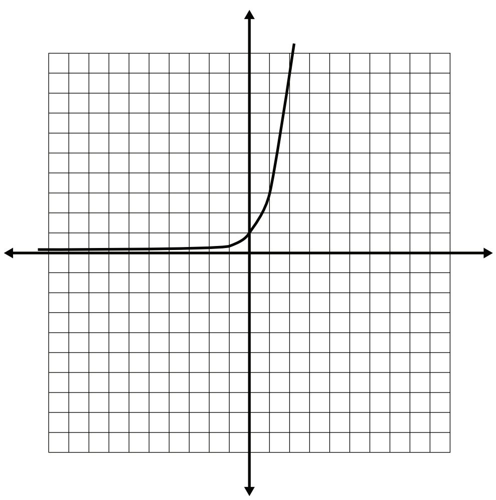
Example 2 Draw each of the following graphs without technology. State the domain, range, and asymptote equation.

Graphing Logarithms

The exponential function corresponding to the base is provided as a convenience.

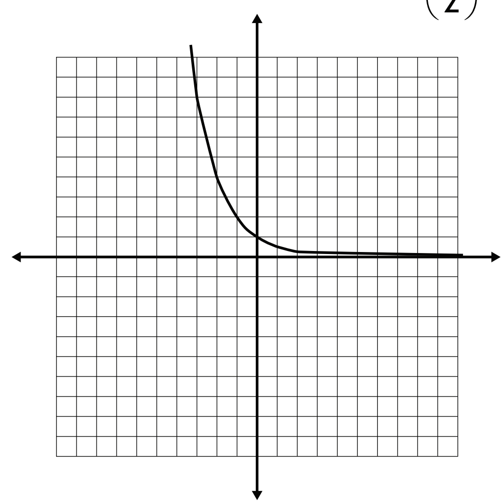
a) $y = \log_3 x$

Given: $y = 3^x$



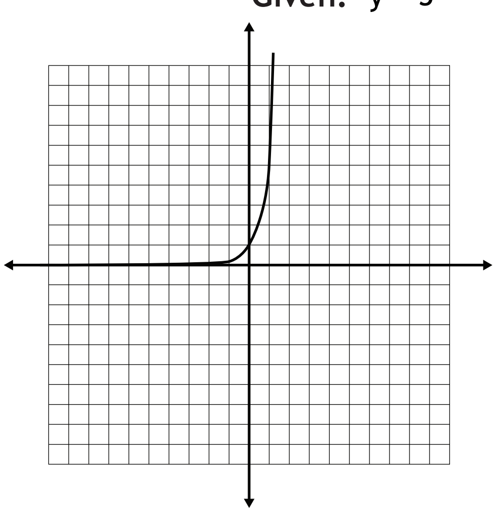
b) $y = \log_{\frac{1}{2}} x$

Given: $y = \left(\frac{1}{2}\right)^x$



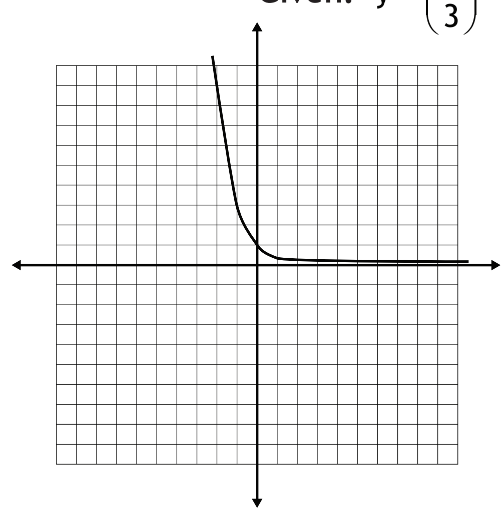
c) $y = \log_5 x$

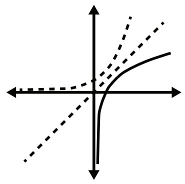
Given: $y = 5^x$



d) $y = \log_{\frac{1}{3}} x$

Given: $y = \left(\frac{1}{3}\right)^x$





$y = \log_b x$

Example 3

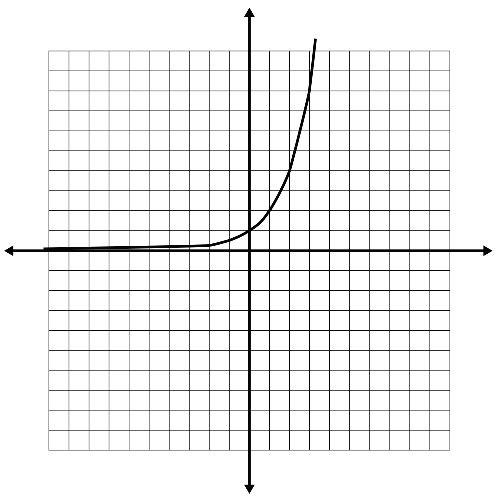
Draw each of the following graphs without technology. State the domain, range, and asymptote equation.

Stretches and Reflections

The exponential function corresponding to the base is provided as a convenience.

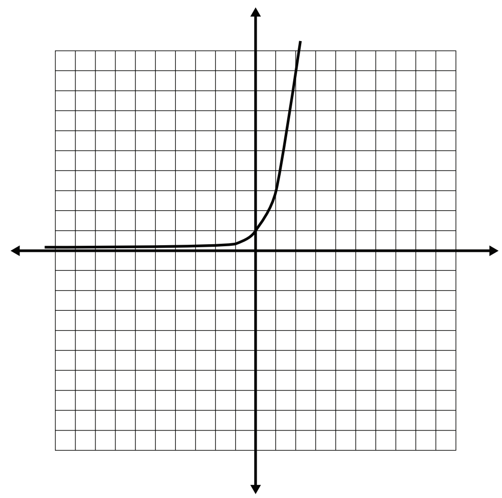
a) $y = 2\log_2 x$

Given: $y = 2^x$



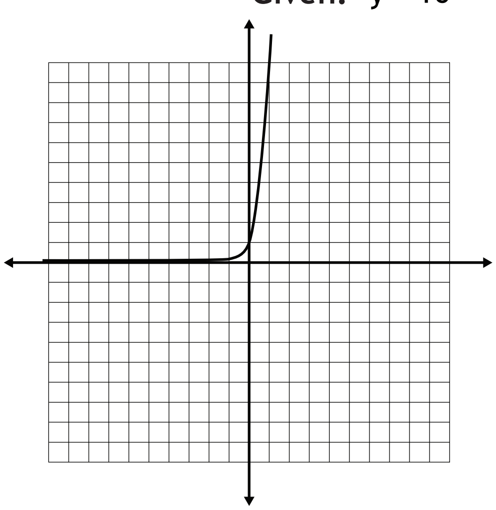
b) $y = -\frac{1}{3}\log_3 x$

Given: $y = 3^x$



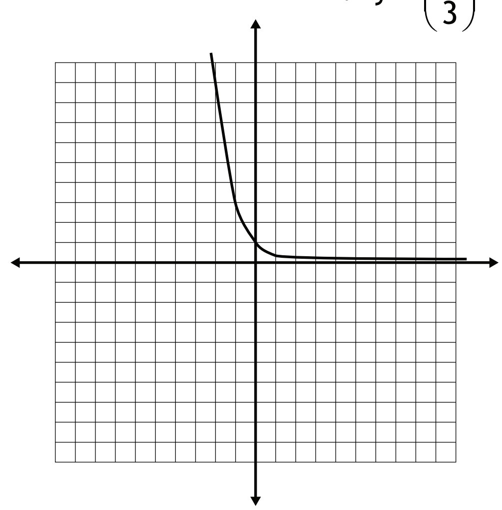
c) $y = \log(2x)$

Given: $y = 10^x$

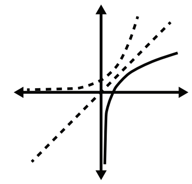


d) $y = \log_{\frac{1}{3}}\left(\frac{1}{2}x\right)$

Given: $y = \left(\frac{1}{3}\right)^x$



Lesson Notes



$y = \log_b x$

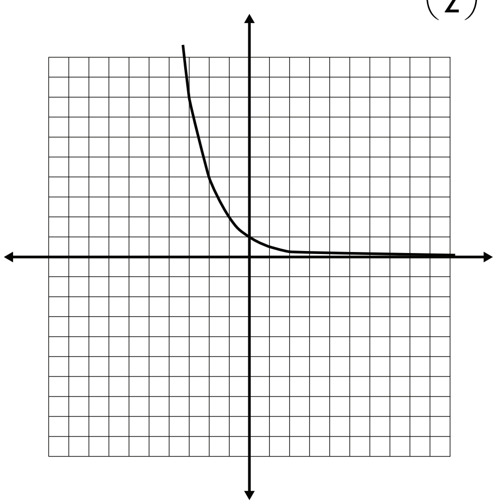
Example 4 Draw each of the following graphs without technology. State the domain, range, and asymptote equation.

Translations

The exponential function corresponding to the base is provided as a convenience.

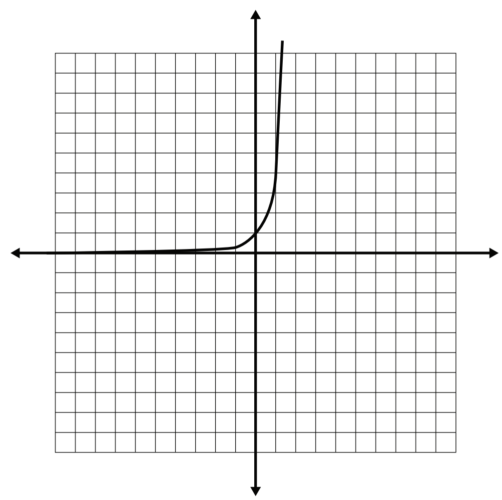
a) $y = \log_{\frac{1}{2}} x - 1$

Given: $y = \left(\frac{1}{2}\right)^x$



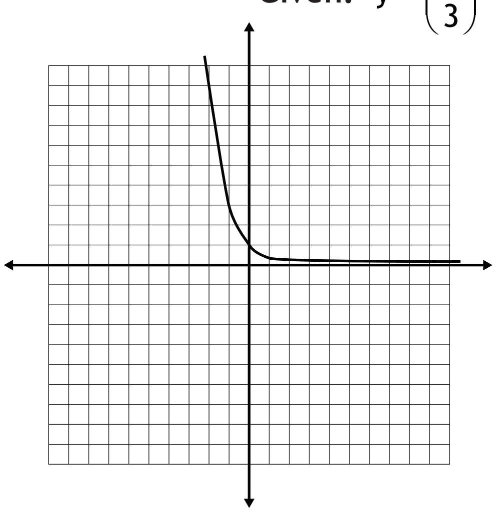
b) $y = \log_4 (x + 2)$

Given: $y = 4^x$



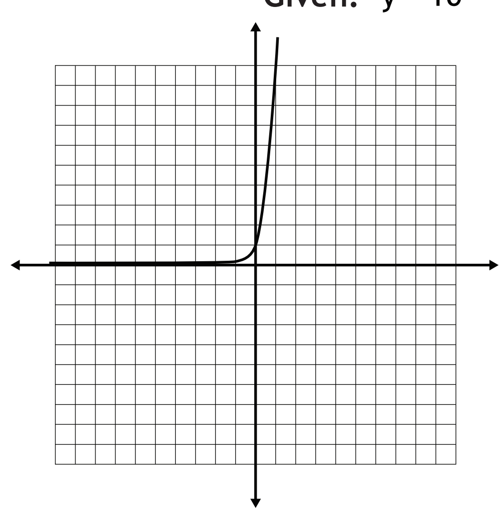
c) $y = \log_{\frac{1}{3}} (x - 3) - 1$

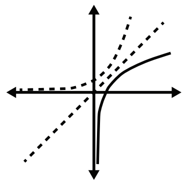
Given: $y = \left(\frac{1}{3}\right)^x$



d) $y = \log(x + 4) + 2$

Given: $y = 10^x$





$y = \log_b x$

Exponential and Logarithmic Functions
LESSON THREE - *Logarithmic Functions*
Lesson Notes

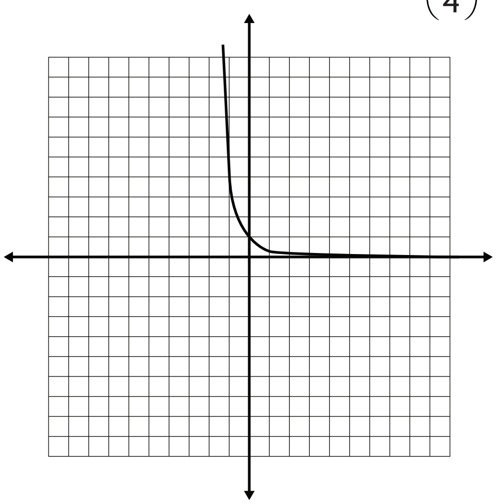
Example 5 Draw each of the following graphs without technology. State the domain, range, and asymptote equation.

Combined Transformations

The exponential function corresponding to the base is provided as a convenience.

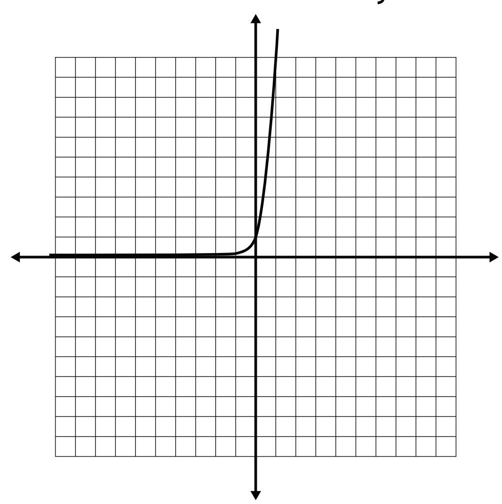
a) $y = \frac{1}{2}\log_{\frac{1}{4}}\left(x+3\right)$

Given: $y = \left(\frac{1}{4}\right)^x$



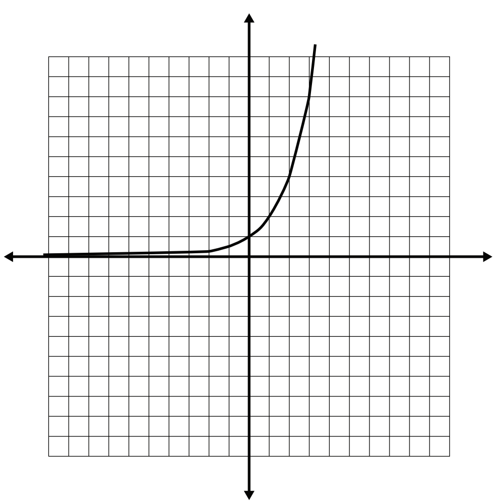
b) $y = -5\log x - 3$

Given: $y = 10^x$



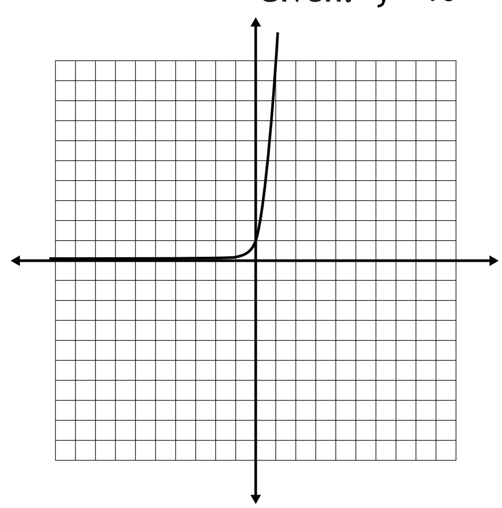
c) $y = 2\log_2\left(2x+6\right) - 1$

Given: $y = 2^x$



d) $y = 10\log\left(x+2\right) - 2$

Given: $y = 10^x$



Lesson Notes

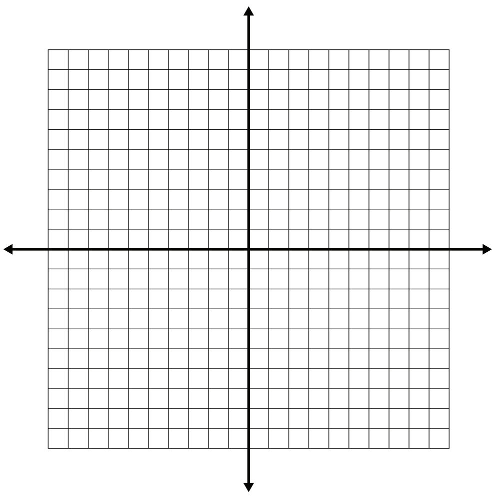
$y = \log_b x$

Example 6

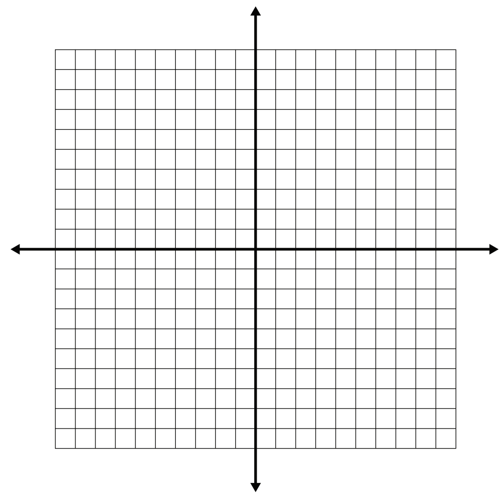
Draw each of the following graphs without technology.
State the domain, range, and asymptote equation.

Other Logarithmic Functions

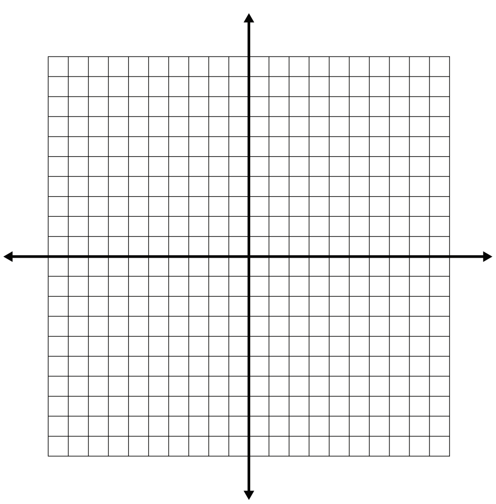
a) $y = \log_2 \sqrt{x}$



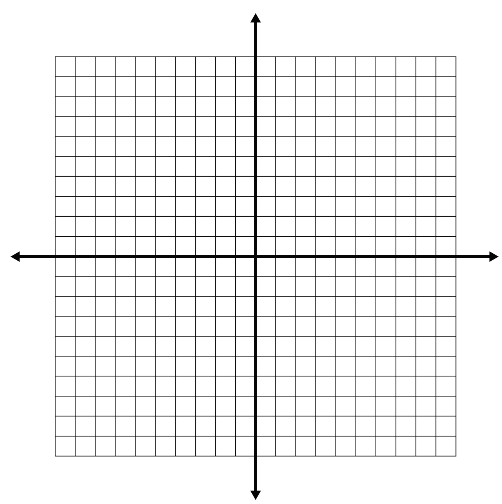
b) $y = \log(x-1)^5$



c) $y = \log_3\left(x^2-4\right) - \log_3(x-2)$



d) $y = \log_9 x + \log_3 x$



Example 7

Solve each equation by (i) finding a common base (if possible), (ii) using logarithms, and (iii) graphing.

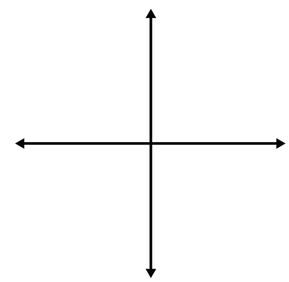
Exponential Equations
(solve multiple ways)

a) $8^{x-2} = 4^{x+1}$

i) Common Base

ii) Solve with Logarithms

iii) Solve Graphically

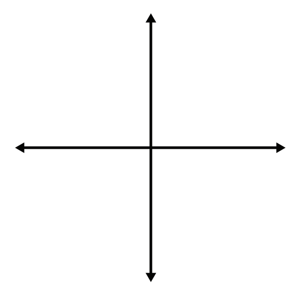


b) $5^{2x+1} = 3^{\frac{x}{2}}$

i) Common Base

ii) Solve with Logarithms

iii) Solve Graphically

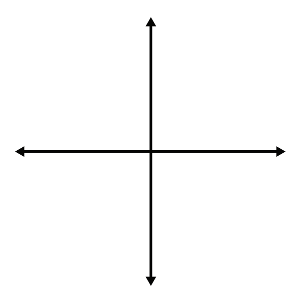


c) $5 = 2^{x-2} + 11$

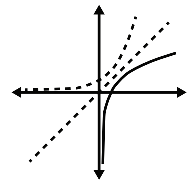
i) Common Base

ii) Solve with Logarithms

iii) Solve Graphically



Exponential and Logarithmic Functions
LESSON THREE- ***Logarithmic Functions***
Lesson Notes



$y = \log_b x$

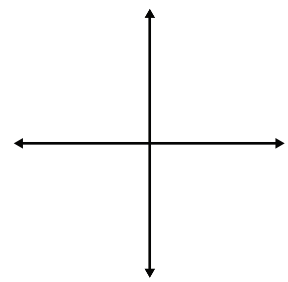
Example 8 Solve each equation by (i) using logarithm laws, and (ii) graphing.

Logarithmic Equations
(solve multiple ways)

a) $\log_3(x+1) = 2$

i) Solve with Logarithm Laws

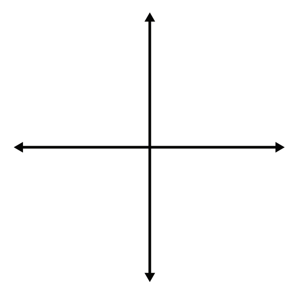
ii) Solve Graphically



b) $\log_5 x^2 + 4\log_5 x = 12$

i) Solve with Logarithm Laws

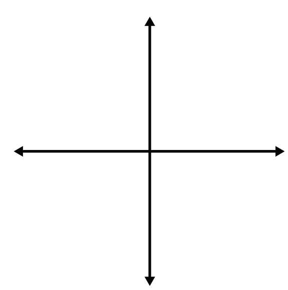
ii) Solve Graphically



c) $\log_2(x-3) + \log_2(x+4) = 3$

i) Solve with Logarithm Laws

ii) Solve Graphically



Example 9

Answer the following questions.

Assorted Mix I

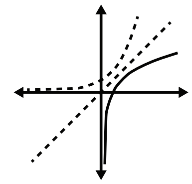
a) The graph of $y = \log_b x$ passes through the point (8, 2). What is the value of b?

b) What are the x- and y-intercepts of $y = \log_2(x + 4)$?

c) What is the equation of the asymptote for $y = \log_3(3x - 8)$?

d) The point (27, 3) lies on the graph of $y = \log_b x$. If the point (4, k) exists on the graph of $y = b^x$, then what is the value of k?

e) What is the domain of $f(x) = \log_x(6 - x)$?



$y = \log_b x$

Example 10 Answer the following questions.

Assorted Mix II

a) The graph of $y = \log_3 x$ can be transformed to the graph of $y = \log_3(9x)$ by either a stretch or a translation. What are the two transformation equations?

b) If the point (4, 1) exists on the graph of $y = \log_4 x$, what is the point after the transformation $y = \log_4(2x + 6)$?

c) A vertical translation is applied to the graph of $y = \log_3 x$ so the image has an x-intercept of (9, 0). What is the transformation equation?

d) What is the point of intersection of $f(x) = \log_2 x$ and $g(x) = \log_2(x + 3) - 2$?

e) What is the x-intercept of $y = a\log_b(kx)$?

Example 11

Answer the following questions.

Assorted Mix III

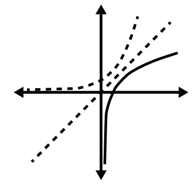
a) What is the equation of the reflection line for the graphs of $f(x) = b^x$ and $g(x) = \left(\frac{1}{b}\right)^x$?

b) If the point (a, 0) exists on the graph of f(x), and the point (0, a) exists on the graph of g(x), what is the transformation equation?

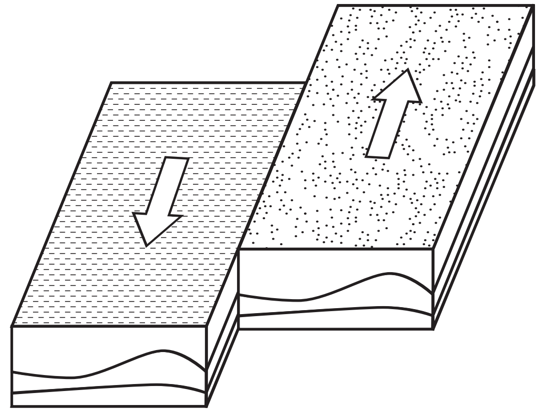
c) What is the inverse of $f(x) = 3^x + 4$?

d) If the graph of $f(x) = \log_4 x$ is transformed by the equation $y = f(3x - 12) + 2$, what is the new domain of the graph?

e) The point (k, 3) exists on the inverse of $y = 2^x$. What is the value of k?



Example 12



The strength of an earthquake is calculated using Richter's formula:

$$\boxed{M = \log\frac{A}{A_0}}$$

where M is the magnitude of the earthquake (unitless), A is the seismograph amplitude of the earthquake being measured (m), and A_0 is the seismograph amplitude of a threshold earthquake (10^{-6} m).

a) An earthquake has a seismograph amplitude of 10^{-2} m.
What is the magnitude of the earthquake?

b) The magnitude of an earthquake is 5.0 on the Richter scale.
What is the seismograph amplitude of this earthquake?

c) Two earthquakes have magnitudes of 4.0 and 5.5.
Calculate the seismograph amplitude ratio for the two earthquakes.

d) The calculation in part (c) required multiple steps because we are comparing each amplitude with A_0, instead of comparing the two amplitudes to each other. It is possible to derive the formula:

$$\boxed{\frac{A_2}{A_1} = 10^{M_2 - M_1}}$$

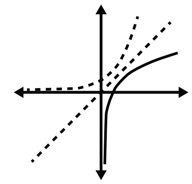
which compares two amplitudes directly without requiring A_0.
Derive this formula.

e) What is the ratio of seismograph amplitudes for earthquakes with magnitudes of 5.0 and 6.0?

f) Show that an equivalent form of the equation is: $\boxed{M_2 - M_1 = \log \frac{A_2}{A_1}}$

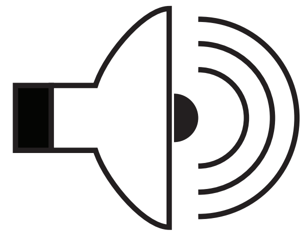
g) What is the magnitude of an earthquake with triple the seismograph amplitude of a magnitude 5.0 earthquake?

h) What is the magnitude of an earthquake with one-fourth the seismograph amplitude of a magnitude 6.0 earthquake?



$y = \log_b x$

Example 13



The loudness of a sound is measured in decibels, and can be calculated using the formula:

$$\boxed{L = 10\log\frac{I}{I_0}}$$

where L is the perceived loudness of the sound (dB), I is the intensity of the sound being measured (W/m^2), and I_0 is the intensity of sound at the threshold of human hearing (10^{-12} W/m^2).

a) The sound intensity of a person speaking in a conversation is 10^{-6} W/m^2. What is the perceived loudness?

b) A rock concert has a loudness of 110 dB. What is the sound intensity?

c) Two sounds have decibel measurements of 85 dB and 105 dB. Calculate the intensity ratio for the two sounds.

d) The calculation in part (c) required multiple steps because we are comparing each sound with I_0, instead of comparing the two sounds to each other. It is possible to derive the formula:

$$\boxed{\frac{I_2}{I_1} = 10^{\frac{L_2 - L_1}{10}}}$$

which compares two sounds directly without requiring I_0. Derive this formula.

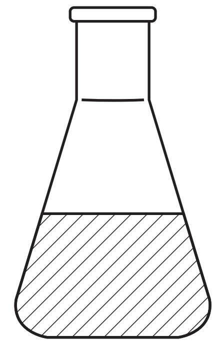
e) How many times more intense is 40 dB than 20 dB?

f) Show that an equivalent form of the equation is: $\boxed{L_2 - L_1 = 10\log\frac{I_2}{I_1}}$

g) What is the loudness of a sound twice as intense as 20 dB?

h) What is the loudness of a sound half as intense as 40 dB?

Example 14



The pH of a solution can be measured with the formula

$$\boxed{pH = -\log\left[H^+\right]}$$

where $[H^+]$ is the concentration of hydrogen ions in the solution (mol/L). Solutions with a pH less than 7 are acidic, and solutions with a pH greater than 7 are basic.

a) What is the pH of a solution with a hydrogen ion concentration of 10^{-4} mol/L? Is this solution acidic or basic?

b) What is the hydrogen ion concentration of a solution with a pH of 11?

c) Two acids have pH values of 3.0 and 6.0. Calculate the hydrogen ion ratio for the two acids.

d) The calculation in part (c) required multiple steps. Derive the formulae *(on right)* that can be used to compare the two acids directly.

$$\frac{\left[H^+\right]_2}{\left[H^+\right]_1} = 10^{-\left(pH_2 - pH_1\right)}$$

and

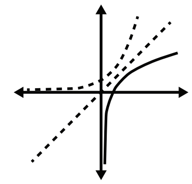
$$pH_2 - pH_1 = -\log\frac{\left[H^+\right]_2}{\left[H^+\right]_1}$$

e) What is the pH of a solution 1000 times more acidic than a solution with a pH of 5?

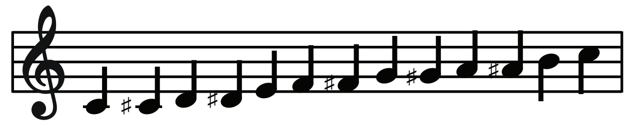
f) What is the pH of a solution with one-tenth the acidity of a solution with a pH of 4?

g) How many times more acidic is a solution with a pH of 2 than a solution with a pH of 4?

Lesson Notes



$y = \log_b x$



Example 15

In music, a chromatic scale divides an octave into 12 equally-spaced pitches. An octave contains 1200 cents *(a unit of measure for musical intervals)*, and each pitch in the chromatic scale is 100 cents apart. The relationship between cents and note frequency is given by the formula:

$$\boxed{c_2 - c_1 = 1200\left(\log_2 \frac{f_2}{f_1}\right)}$$

a) How many cents are in the interval between A (440 Hz) and B (494 Hz)?

b) There are 100 cents between F# and G. If the frequency of F# is 740 Hz, what is the frequency of G?

c) How many cents separate two notes, where one note is double the frequency of the other note?

For more practice solving logarithmic equations, return to ***Exponential Functions*** and solve the word problems using logarithms.

Polynomial, Radical, and Rational Functions Lesson One: Polynomial Functions

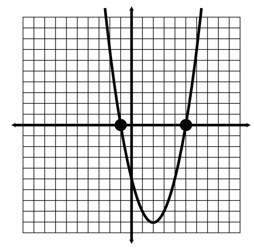
Example 1: a) Leading coefficient is a_n ; polynomial degree is n; constant term is a_0. i) 3; 1; -2 ii) 1; 3; -1 iii) 5; 0; 5

b) i) Y ii) N iii) Y iv) N v) Y vi) N vii) N viii) Y ix) N

Example 2: a) i) Even-degree polynomials with a positive leading coefficient have a trendline that matches an upright parabola. End behaviour: The graph starts in the upper-left quadrant (II) and ends in the upper-right quadrant (I).
ii) Even-degree polynomials with a negative leading coefficient have a trendline that matches an upside-down parabola. End behaviour: The graph starts in the lower-left quadrant (III) and ends in the lower-right quadrant (IV).

b) i) Odd-degree polynomials with a positive leading coefficient have a trendline matching the line y = x. The end behaviour is that the graph starts in the lower-left quadrant (III) and ends in the upper-right quadrant (I).
ii) Odd-degree polynomials with a negative leading coefficient have a trendline matching the line y = -x. The end behaviour is that the graph starts in the upper-left quadrant (II) and ends in the lower-right quadrant (IV).

Example 3: a) Zero of a Polynomial Function: Any value of x that satisfies the equation P(x) = 0 is called a zero of the polynomial. A polynomial can have several unique zeros, duplicate zeros, or no real zeros. i) Yes; P(-1) = 0 ii) No; P(3) ≠ 0.
b) Zeros: -1, 5.
c) The x-intercepts of the polynomial's graph are -1 and 5. These are the same as the zeros of the polynomial.
d) "Zero" describes a property of a function; "Root" describes a property of an equation; and "x-intercept" describes a property of a graph.

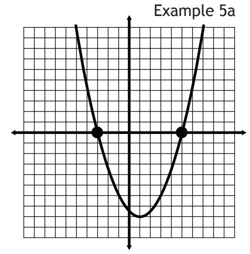


Example 4: a) Multiplicity of a Zero: The multiplicity of a zero (or root) is how many times the root appears as a solution. Zeros give an indication as to how the graph will behave near the x-intercept corresponding to the root.
b) Zeros: -3 (multiplicity 1) and 1 (multiplicity 1). **c)** Zero: 3 (multiplicity 2).
d) Zero: 1 (multiplicity 3). **e)** Zeros: -1 (multiplicity 2) and 2 (multiplicity 1).

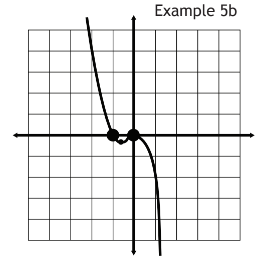
Example 5: a) i) Zeros: -3 (multiplicity 1) and 5 (multiplicity 1).
ii) y-intercept: (0, -7.5). iii) End behaviour: graph starts in QII, ends in QI.
iv) Other points: parabola vertex (1, -8).

b) i) Zeros: -1 (multiplicity 1) and 0 (multiplicity 2). ii) y-intercept: (0, 0).
iii) End behaviour: graph starts in QII, ends in QIV.
iv) Other points: (-2, 4), (-0.67, -0.15), (1, -2).

Example 5a



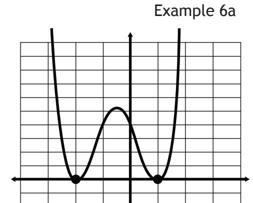
Example 5b



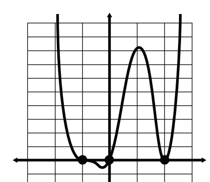
Example 6: a) i) Zeros: -2 (multiplicity 2) and 1 (multiplicity 2).
ii) y-intercept: (0, 4). iii) End behaviour: graph starts in QII, ends in QI.
iv) Other points: (-3, 16), (-0.5, 5.0625), (2, 16).

b) i) Zeros: -1 (multiplicity 3), 0 (multiplicity 1), and 2 (multiplicity 2).
ii) y-intercept: (0, 0). iii) End behaviour: graph starts in QII, ends in QI.
iv) Other points: (-2, 32), (-0.3, -0.5), (1.1, 8.3), (3, 192).

Example 6a



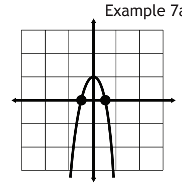
Example 6b



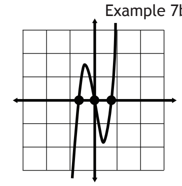
Example 7: a) i) Zeros: -0.5 (multiplicity 1) and 0.5 (multiplicity 1).
ii) y-intercept: (0, 1). iii) End behaviour: graph starts in QIII, ends in QIV.
iv) Other points: parabola vertex (0, 1).

b) i) Zeros: -0.67 (multiplicity 1), 0 (multiplicity 1), and 0.75 (multiplicity 1).
ii) y-intercept: (0, 0). iii) End behaviour: graph starts in QIII, ends in QI.
iv) Other points: (-1, -7), (-0.4, 1.5), (0.4, -1.8), (1, 5).

Example 7a



Example 7b



Example 8:

a) $P(x) = -\frac{1}{3}(x+3)(x-4)$

b) $P(x) = \frac{1}{8}(x+2)^3(x-1)$

Example 9:

a) $P(x) = -\frac{1}{12}x^3(x-5)^2$

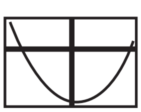
b) $P(x) = \frac{1}{32}(x+6)(x+2)(x-2)(x-6)$

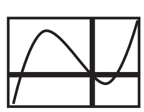
Example 10:

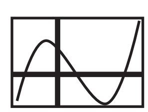
a) $P(x) = \frac{1}{2}(2x+3)(3x-4)$

b) $P(x) = \frac{1}{288}(x+6)^2(3x+8)(4x-9)$

Example 11: **a)** x: [-15, 15, 1], y: [-169, 87, 1] **b)** x: [-12, 7, 1], y: [-192, 378, 1] **c)** x: [-12, 24, 1], y: [-1256, 2304, 1]

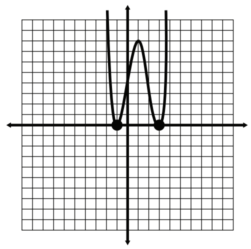




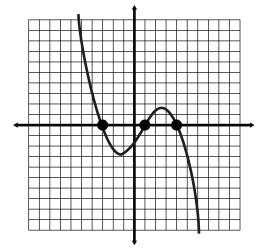


Answer Key

Example 12: a) $P(x)=\frac{1}{2}(x+1)^2(x-3)^2$

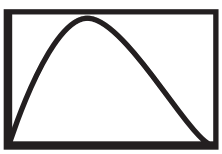


b) $P(x)=-\frac{1}{8}(x+3)(x-1)(x-4)$



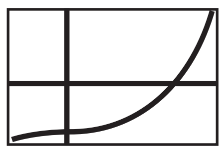
Example 13:

a) $V(x) = x(20 - 2x)(16 - 2x)$
b) $0 < x < 8$ or $(0, 8)$
c) Window Settings:
x: [0,8, 1], y: [0, 420, 1]
d) When the side length of a corner square is 2.94 cm, the volume of the box will be maximized at 420.11 cm^3.
e) The volume of the box is greater than 200 cm^3 when $0.74 < x < 5.93$. or $(0.74, 5.93)$



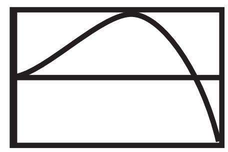
Example 14:

a) $P_{product}(x) = x^2(x + 2)$; $P_{sum}(x) = 3x + 2$
b) $x^3 + 2x^2 - 3x - 11550 = 0$.
c) Window Settings:
x: [-10, 30, 1], y: [-12320, 17160, 1]
Quinn and Ralph are 22 since x = 22.
Audrey is two years older, so she is 24.



Example 15:

a) Window Settings:
x: [0, 6, 1], y: [-1.13, 1.17, 1]
b) At 3.42 seconds, the maximum volume of 1.17 L is inhaled
c) One breath takes 5.34 seconds to complete.
d) 64% of the breath is spent inhaling.



Example 16: $V(h)=\frac{1}{4}\pi\left(64h-h^3\right)$

Polynomial, Radical, and Rational Functions Lesson Two: Polynomial Division

Example 1: a) Quotient: $x^2 - 5$; R = 4 **b)** P(x): $x^3 + 2x^2 - 5x - 6$; $D(x) = x + 2$; $Q(x) = x^2 - 5$; R = 4 **c)** L.S.= R.S. **d)** $Q(x) = x^2 - 5 + 4/(x + 2)$ **e)** $Q(x) = x^2 - 5 + 4/(x + 2)$

Example 2: a) $3x^2 - 7x + 9 - 10/(x + 1)$ **b)** $x^2 + 2x + 1$ **c)** $x^2 - 2x + 4 - 9/(x + 2)$

Example 3: a) $3x^2 + 3x + 2 - 1/(x - 1)$ **b)** $3x^3 - x^2 + 2x - 1$ **c)** $2x^3 + 2x^2 - 5x - 5 - 1/(x - 1)$

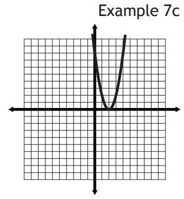
Example 4: a) $x - 2$ **b)** 2 **c)** $x^2 - 4$ **d)** $x^2 + 5x + 12 + 36/(x - 3)$

Example 5: a) a = -5 **b)** a = -5

Example 6: The dimensions of the base are x + 5 and x - 3

Example 7: a) $f(x) = 2(x + 1)(x - 2)^2$ **b)** $g(x) = x + 1$ **c)** $Q(x) = 2(x - 2)^2$

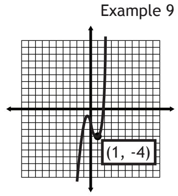
Example 7c



Example 8: a) $f(x) = 4x^3 - 7x - 3$ **b)** $g(x) = x - 1$

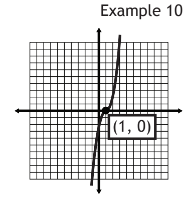
Example 9: a) R = -4
b) R = -4. The point (1, -4) exists on the graph. The remainder is just the y-value of the graph when x = 1.
c) Both synthetic division and the remainder theorem return a result of -4 for the remainder.
d) i) R = 4 ii) R = -2 iii) R = -2
e) When the polynomial P(x) is divided by x - a, the remainder is P(a).

Example 9

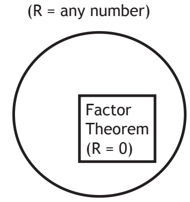


Example 10: a) R = 0
b) R = 0. The point (1, 0) exists on the graph. The remainder is just the y-value of the graph.
c) Both synthetic division and the remainder theorem return a result of 0 for the remainder.
d) If P(x) is divided by x - a, and P(a) = 0, then x - a is a factor of P(x).
e) When we use the remainder theorem, the result can be any real number. If we use the remainder theorem and get a result of zero, the factor theorem gives us one additional piece of information - the divisor fits evenly into the polynomial and is therefore a factor of the polynomial. Put simply, we're always using the remainder theorem, but in the special case of R = 0 we get extra information from the factor theorem.

Example 10



Remainder Theorem
(R = any number)



Example 11: a) $P(-1) \neq 0$, so x + 1 is not a factor.
b) $P(-2) \neq 0$, so x + 2 is not a factor.
c) $P(1/3) = 0$, so 3x - 1 is a factor.
d) $P(-3/2) \neq 0$, so 2x + 3 is not a factor.
Example 12: a) k = 3 **b)** k = -7 **c)** k = -7 **d)** k = -5
Example 13: m = 4 and n = -7
Example 14: m = 4 and n = -3
Example 15: a = 5

Interval Notation

Math 30-1 students are expected to know that domain and range can be expressed using ***interval notation***.

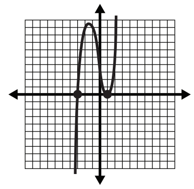
() - Round Brackets: Exclude point from interval.
[] - Square Brackets: Include point in interval.
Infinity ∞ always gets a round bracket.

Examples: $x \geq -5$ becomes $[-5, \infty)$;
$1 < x \leq 4$ becomes $(1, 4]$;
$x \in R$ becomes $(-\infty, \infty)$;
$-8 \leq x < 2$ or $5 \leq x < 11$ becomes $[-8, 2) \cup [5, 11)$,
where U means "or", or ***union of sets***;
$x \in R, x \neq 2$ becomes $(-\infty, 2) \cup (2, \infty)$;
$-1 \leq x \leq 3, x \neq 0$ becomes $[-1, 0) \cup (0, 3]$.

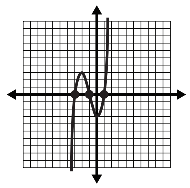
Answer Key

Polynomial, Radical, and Rational Functions Lesson Three: Polynomial Factoring

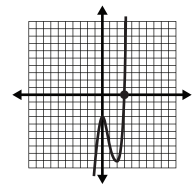
Example 1: a) The integral factors of the constant term of a polynomial are potential zeros of the polynomial.
b) Potential zeros of the polynomial are ±1 and ±3.
c) The zeros of P(x) are -3 and 1 since P(-3) = 0 and P(1) = 0
d) The x-intercepts match the zeros of the polynomial
e) $P(x) = (x + 3)(x - 1)^2$.



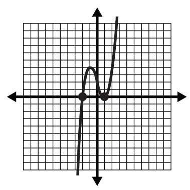
Example 2: a) $P(x) = (x + 3)(x + 1)(x - 1)$.
b) All of the factors can be found using the graph.
c) Factor by grouping.



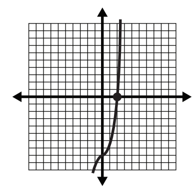
Example 3: a) $P(x) = (2x^2 + 1)(x - 3)$.
b) Not all of the factors can be found using the graph.
c) Factor by grouping.



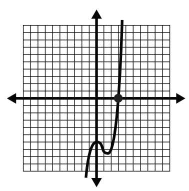
Example 4: a) $P(x) = (x + 2)(x - 1)^2$.
b) All of the factors can be found using the graph.
c) No.



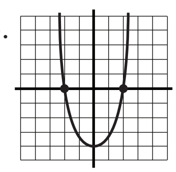
Example 5: a) $P(x) = (x^2 + 2x + 4)(x - 2)$.
b) Not all of the factors can be found using the graph.
c) $x^3 - 8$ is a difference of cubes



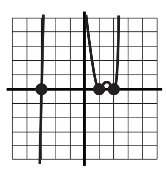
Example 6: a) $P(x) = (x^2 + x + 2)(x - 3)$.
b) Not all of the factors can be found using the graph.
c) No.



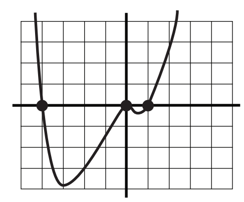
Example 7: a) $P(x) = (x^2 + 4)(x - 2)(x + 2)$.
b) Not all of the factors can be found using the graph.
c) $x^4 - 16$ is a difference of squares.



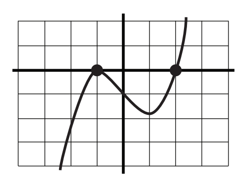
Example 8: a) $P(x) = (x + 3)(x - 1)^2(x - 2)^2$.
b) All of the factors can be found using the graph.
c) No.



Example 9:
a) $P(x) = 1/2x^2(x + 4)(x - 1)$.



b) $P(x) = 2(x + 1)^2(x - 2)$.



Example 10: Width = 10 cm; Height = 7 cm; Length = 15 cm

Example 11: -8; -7; -6

Example 12: k = 2; P(x) = (x + 3)(x - 2)(x - 6)

Example 13: a = -3 and b = -1

Example 14: a) x = -3, 2, and 4 **b)** $x = \frac{-5 - \sqrt{37}}{6}, -1, \frac{-5 + \sqrt{37}}{6}$

Quadratic Formula

From Math 20-1:
The roots of a quadratic equation with the form $ax^2 + bx + c = 0$ can be found with the ***quadratic formula***:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Answer Key

Polynomial, Radical, and Rational Functions Lesson Four: Radical Functions

Example 1:

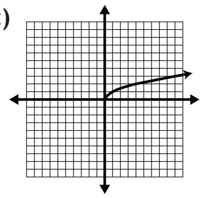
a)

x	f(x)
-1	undefined
0	0
1	1
4	2
9	3

b) Domain: $x \geq 0$; Range: $y \geq 0$

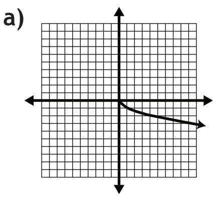
Interval Notation:
Domain: $[0, \infty)$; Range: $[0, \infty)$

c)

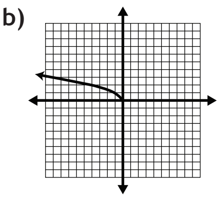


Example 2:

a)

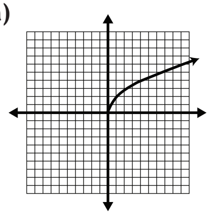


b)

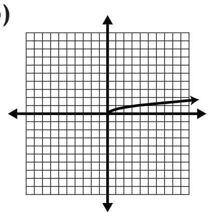


Example 3:

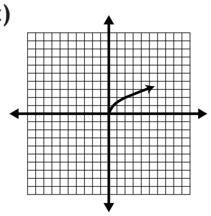
a)



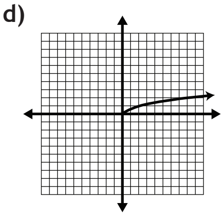
b)



c)

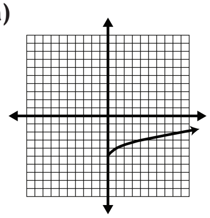


d)

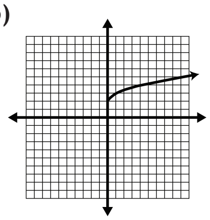


Example 4:

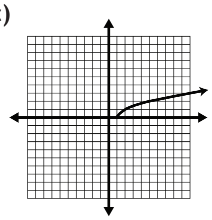
a)



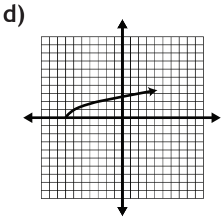
b)



c)

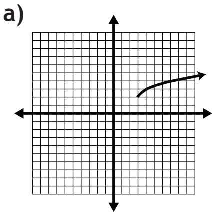


d)

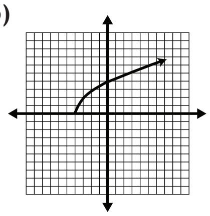


Example 5:

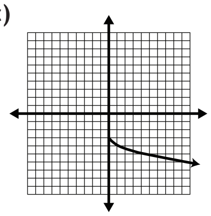
a)



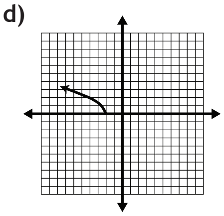
b)



c)

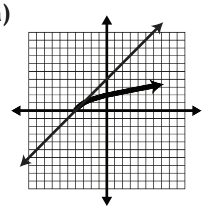


d)



Example 6:

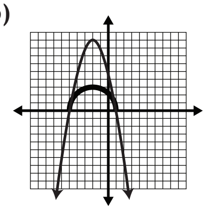
a)



ORIGINAL:
Domain: $x \in R$ or $(-\infty, \infty)$
Range: $y \in R$ or $(-\infty, \infty)$

TRANSFORMED:
Domain: $x \geq -4$ or $[-4, \infty)$
Range: $y \geq 0$ or $[0, \infty)$

b)

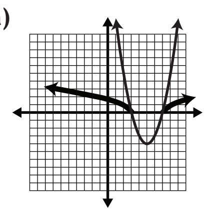


ORIGINAL:
Domain: $x \in R$ or $(-\infty, \infty)$
Range: $y \leq 9$ or $(-\infty, 9]$

TRANSFORMED:
Domain: $-5 \leq x \leq 1$ or $[-5, 1]$
Range: $0 \leq y \leq 3$ or $[0, 3]$

Example 7:

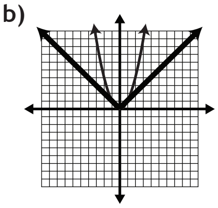
a)



ORIGINAL:
Domain: $x \in R$ or $(-\infty, \infty)$
Range: $y \geq -4$ or $[-4, \infty)$

TRANSFORMED:
Domain: $x \leq 3$ or $x \geq 7$ or $(-\infty, 3] \cup [7, \infty)$
Range: $y \geq 0$ or $[0, \infty)$

b)

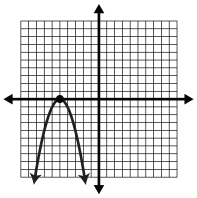


ORIGINAL:
Domain: $x \in R$ or $(-\infty, \infty)$
Range: $y \geq 0$ or $[0, \infty)$

TRANSFORMED:
Domain: $x \in R$ or $(-\infty, \infty)$
Range: $y \geq 0$ or $[0, \infty)$

Example 8:

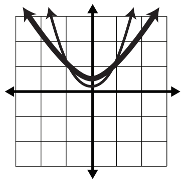
a)



ORIGINAL:
Domain: x ε R or (-∞, ∞)
Range: y ≤ 0 or (-∞, 0]

TRANSFORMED:
Domain: x = -5
Range: y = 0

b)

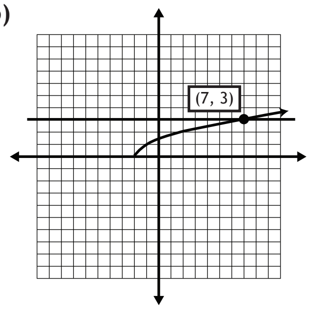


ORIGINAL:
Domain: x ε R or (-∞, ∞)
Range: y ≥ 0.25 or [0.25, ∞)

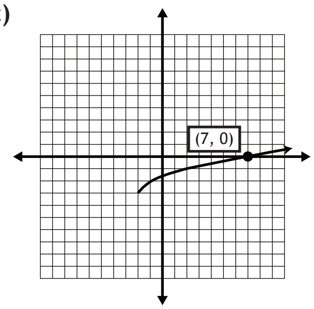
TRANSFORMED:
Domain: x ε R or (-∞, ∞)
Range: y ≥ 0.5 or [0.5, ∞)

Example 9:
a) x = 7

b)

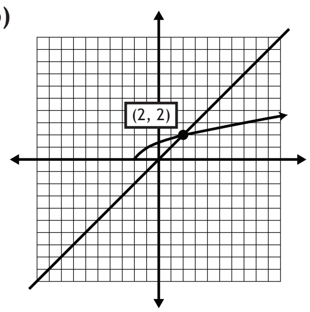


c)

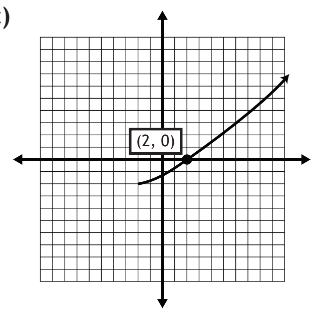


Example 10:
a) x = 2

b)

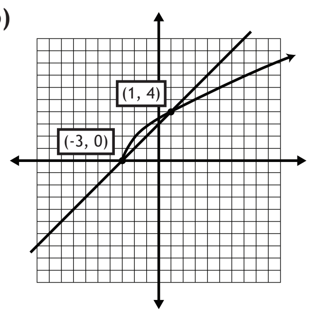


c)

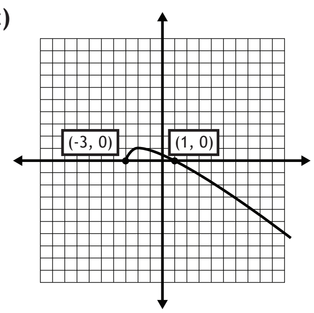


Example 11:
a) x = -3, 1

b)

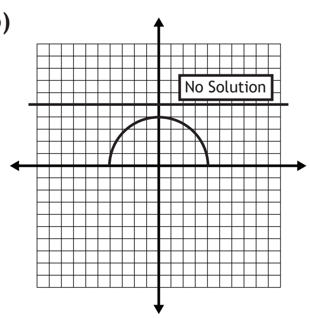


c)

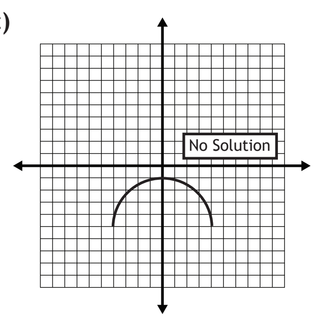


Example 12:
a) No Solution

b)



c)

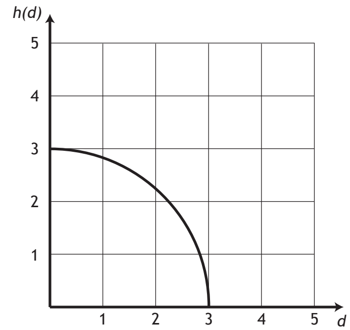


Example 13:

a) $\sqrt{x+2} = 2$ **b)** $\sqrt{x-1} + 2 = -x + 5$ **c)** $-\sqrt{x-4} + 1 = -1$ **d)** $3\sqrt{x} = -3x + 6$

Example 14:

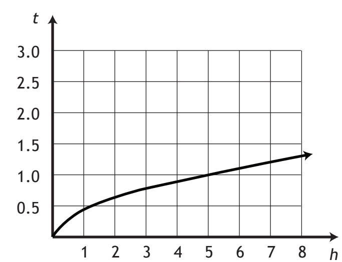
a) $h(d) = \sqrt{9 - d^2}$
b) Domain: 0 ≤ d ≤ 3; Range: 0 ≤ h(d) ≤ 3 or Domain: [0, 3]; Range: [0, 3].
When d = 0, the ladder is vertical.
When d = 3, the ladder is horizontal.
c) 2 m



Example 15:

a) $\sqrt{2}$ × original time
b) 1/2 × original time
c)

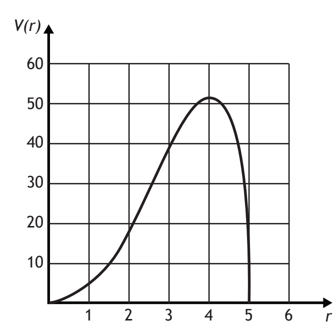
h	t
1	0.4517
4	0.9035
8	1.2778



Example 16:

a) $V(r) = \frac{1}{3}\pi r^2 \sqrt{25 - r^2}$

b)



Answer Key

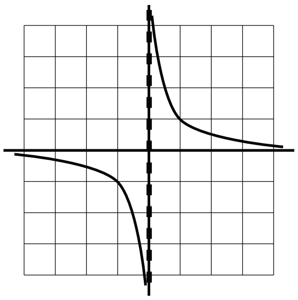
Polynomial, Radical, and Rational Functions Lesson Five: Rational Functions I

Example 1:

a)

x	y
-2	-0.5
-1	-1
-0.5	-2
-0.25	-4
0	undef.
0.25	4
0.5	2
1	1
2	0.5

b)



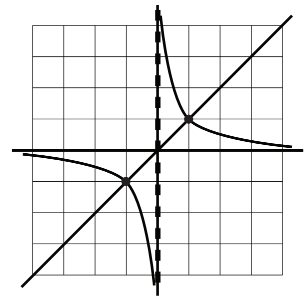
c)

1. The vertical asymptote of the reciprocal graph occurs at the x-intercept of y = x.

2. The invariant points *(points that are identical on both graphs)* occur when y = ±1.

3. When the graph of y = x is below the x-axis, so is the reciprocal graph. When the graph of y = x is above the x-axis, so is the reciprocal graph.

d)

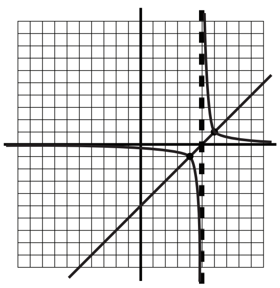


Example 2:

a) Original Graph:
Domain: x ε R or (-∞, ∞);
Range: y ε R or (-∞, ∞)

Reciprocal Graph:
Domain: x ε R, x ≠ 5 or (-∞, 5) U (5, ∞);
Range: y ε R, y ≠ 0 or (-∞, 0) U (0, ∞)

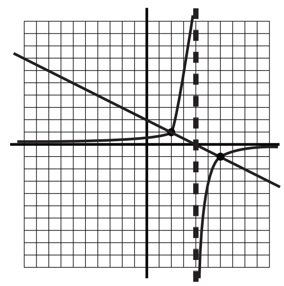
Asymptote Equation(s):
Vertical: x = 5;
Horizontal: y = 0



b) Original Graph:
Domain: x ε R or (-∞, ∞);
Range: y ε R or (-∞, ∞)

Reciprocal Graph:
Domain: x ε R, x ≠ 4 or (-∞, 4) U (4, ∞);
Range: y ε R, y ≠ 0 or (-∞, 0) U (0, ∞)

Asymptote Equation(s):
Vertical: x = 4;
Horizontal: y = 0



Example 3:

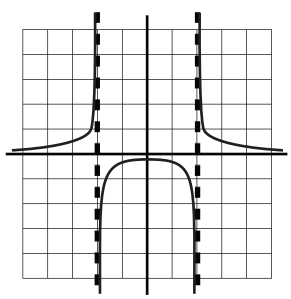
a)

x	y
-3	0.20
-2	undef.
-1	-0.33
0	-0.25
1	-0.33
2	undef.
3	0.20

x	y
-2.05	4.94
-1.95	-5.06

x	y
1.95	-5.06
2.05	4.94

b)



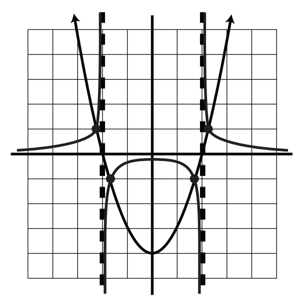
c)

1. The vertical asymptotes of the reciprocal graph occur at the x-intercepts of y = x^2 - 4.

2. The invariant points (points that are identical in both graphs) occur when y = ±1.

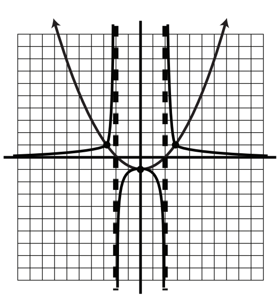
3. When the graph of y = x^2 - 4 is below the x-axis, so is the reciprocal graph.
When the graph of y = x^2 - 4 is above the x-axis, so is the reciprocal graph.

d)

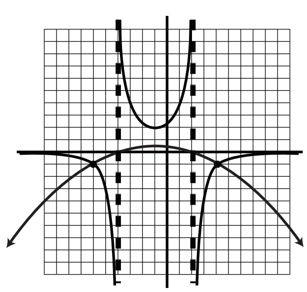


Example 4:

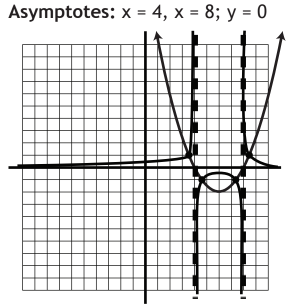
a) Original: x ε R; y ≥ -1
or D: (-∞, ∞); R: [-1, ∞).
Reciprocal: x ε R, x ≠ -2, 2; y ≤ -1 or y > 0
or D: (-∞, -2) U (-2, 2) U (2, ∞); R: (-∞, -1] U (0, ∞)
Asymptotes: x = ±2; y = 0



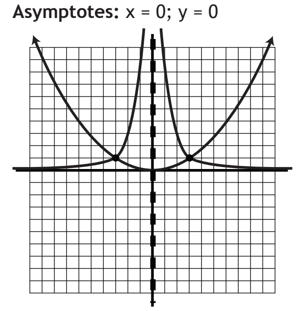
b) Original: x ε R; y ≤ 1/2
or D: (-∞, ∞); R: (-∞, 1/2].
Reciprocal: x ε R, x ≠ -4, 2; y < 0 or y ≥ 2
or D: (-∞, -4) U (-4, 2) U (2, ∞); R: (-∞, 0) U [2, ∞)
Asymptotes: x = -4, x = 2; y = 0



c) Original: x ε R; y ≥ -2
or D: (-∞, ∞); R: [-2, ∞).
Reciprocal: x ε R, x ≠ 4, 8; y ≤ -1/2 or y > 0
or D: (-∞, 4) U (4, 8) U (8, ∞); R: (-∞, -1/2] U (0, ∞)
Asymptotes: x = 4, x = 8; y = 0



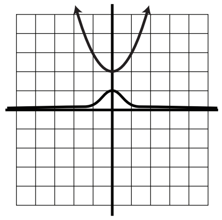
d) Original: x ε R; y ≥ 0
or D: (-∞, ∞); R: [0, ∞).
Reciprocal: x ε R, x ≠ 0; y > 0
or D: (-∞, 0) U (0, ∞); R: (0, ∞)
Asymptotes: x = 0; y = 0



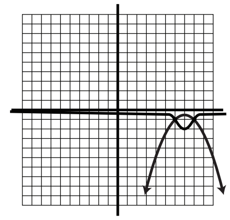
Answer Key

Example 4 (continued):

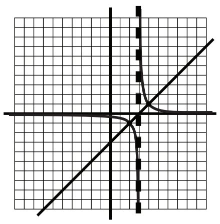
e) Original: x ε R; y ≥ 2
or D: (-∞, ∞); R: [2, ∞).
Reciprocal: x ε R; 0 < y ≤ 1/2
or D: (-∞, ∞); R: (0, 1/2]
Asymptotes: y = 0

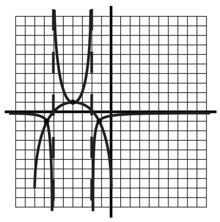


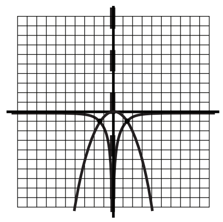
f) Original: x ε R; y ≤ -1/2
or D: (-∞, ∞); R: (-∞, -1/2].
Reciprocal: x ε R; -2 ≤ y < 0
or D: (-∞, ∞); R: [-2, 0)
Asymptotes: y = 0

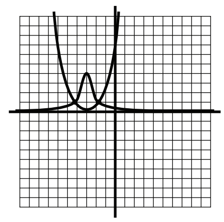


Example 5:

a) 

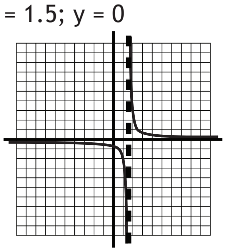
b) 

c) 

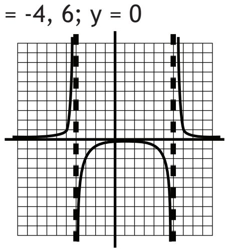
d) 

Example 6:

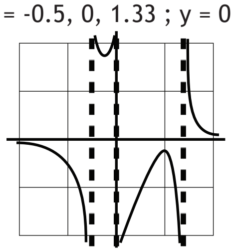
a) x = 1.5; y = 0



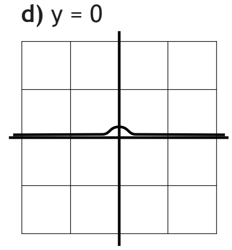
b) x = -4, 6; y = 0



c) x = -0.5, 0, 1.33 ; y = 0

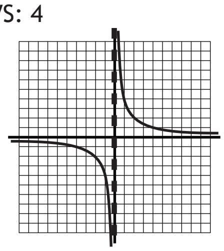


d) y = 0

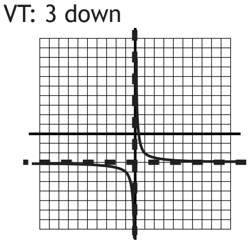


Example 7:

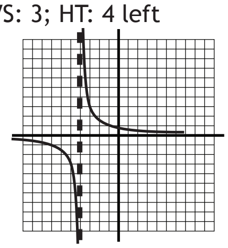
a) VS: 4



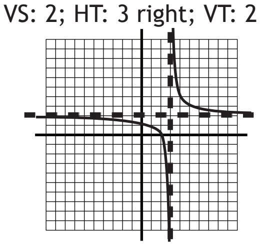
b) VT: 3 down



c) VS: 3; HT: 4 left

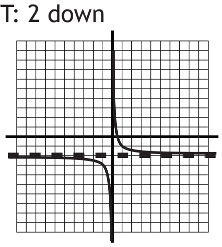


d) VS: 2; HT: 3 right; VT: 2 up

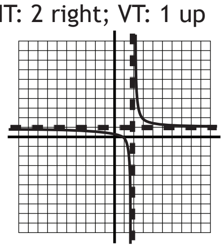


Example 8:

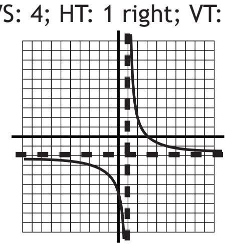
a) VT: 2 down



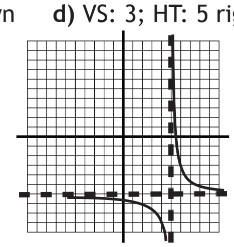
b) HT: 2 right; VT: 1 up



c) VS: 4; HT: 1 right; VT: 2 down



d) VS: 3; HT: 5 right; VT: 2 down



Example 9:

a) P(V) = nRT(1/V).

b) 1/2 × original

c) 2 × original

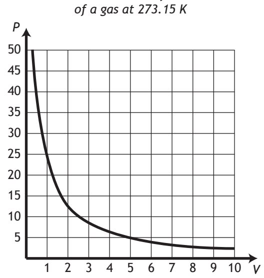
d) 8.3 kPa•L/mol•K

e) See table & graph

f) See table & graph

V (L)	P (kPa)
0.5	50
1.0	25
2.0	12.5
5.0	5.0
10.0	2.5

Pressure V.S. Volume of 0.011 mol of a gas at 273.15 K



Example 10:

a) 1/4 × original

b) 1/9 × original

c) 4 × original

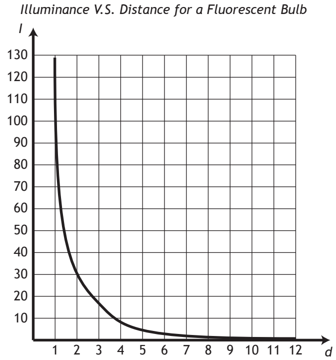
d) 16 × original

e) See table & graph

f) See table & graph

d (m)	I (W/m²)
1	$\frac{400}{\pi}$
2	$\frac{100}{\pi}$
4 ORIGINAL	$\frac{25}{\pi}$
8	$\frac{25}{4\pi}$
12	$\frac{25}{9\pi}$

Illuminance V.S. Distance for a Fluorescent Bulb

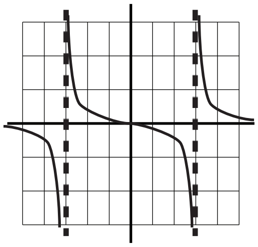


Answer Key

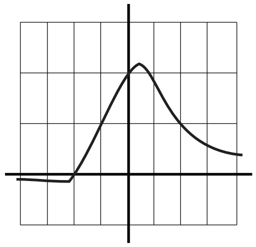
Polynomial, Radical, and Rational Functions Lesson Six: Rational Functions II

Example 1:

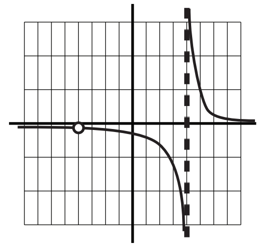
a) $y = \frac{x}{x^2 - 9}$



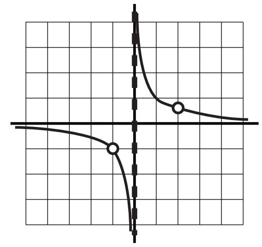
b) $y = \frac{x + 2}{x^2 + 1}$



c) $y = \frac{x + 4}{x^2 - 16}$

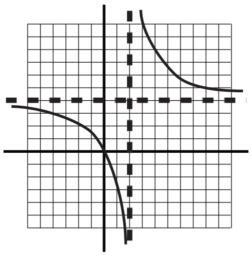


d) $y = \frac{x^2 - x - 2}{x^3 - x^2 - 2x}$

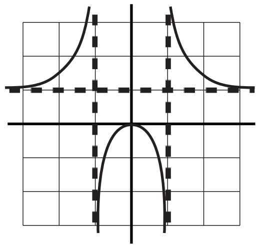


Example 2:

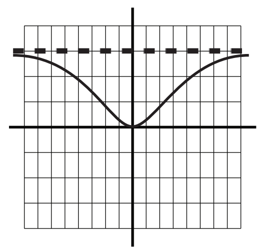
a) $y = \frac{4x}{x - 2}$



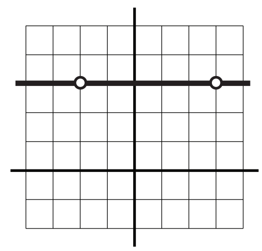
b) $y = \frac{x^2}{x^2 - 1}$



c) $y = \frac{3x^2}{x^2 + 9}$

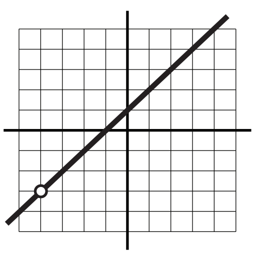


d) $y = \frac{3x^2 - 3x - 18}{x^2 - x - 6}$

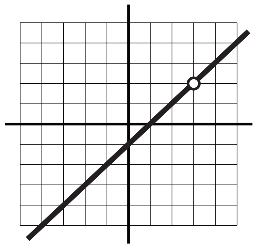


Example 3:

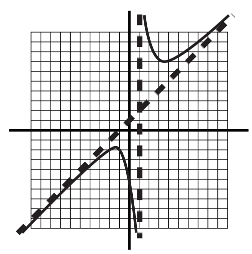
a) $y = \frac{x^2 + 5x + 4}{x + 4}$



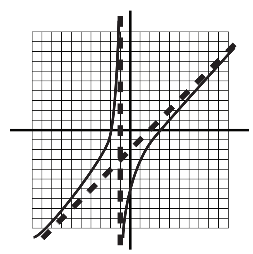
b) $y = \frac{x^2 - 4x + 3}{x - 3}$



c) $y = \frac{x^2 + 5}{x - 1}$



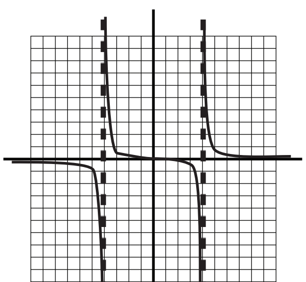
d) $y = \frac{x^2 - x - 6}{x + 1}$



Example 4:

i) Horizontal Asymptote: $y = 0$
ii) Vertical Asymptote(s): $x = \pm 4$
iii) y - intercept: (0, 0)
iv) x - intercept(s): (0, 0)
v) Domain: $x \varepsilon R$, $x \neq \pm 4$;
Range: $y \varepsilon R$

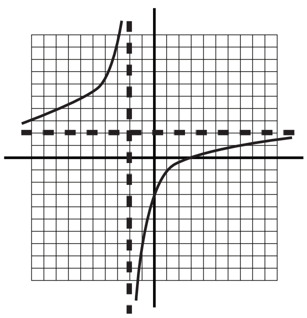
or D: $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$; R: $(-\infty, \infty)$



Example 5:

i) Horizontal Asymptote: $y = 2$
ii) Vertical Asymptote(s): $x = -2$
iii) y - intercept: (0, -3)
iv) x - intercept(s): (3, 0)
v) Domain: $x \varepsilon R$, $x \neq -2$;
Range: $y \varepsilon R$, $y \neq 2$

or D: $(-\infty, -2) \cup (-2, \infty)$; R: $(-\infty, 2) \cup (2, \infty)$

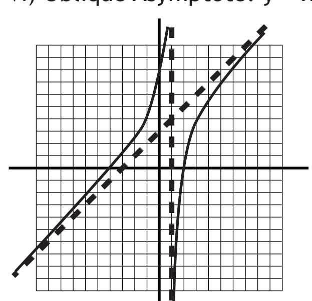


Example 6:

i) Horizontal Asymptote: None
ii) Vertical Asymptote(s): $x = 1$
iii) y - intercept: (0, 8)
iv) x - intercept(s): (-4, 0), (2, 0)
v) Domain: $x \varepsilon R$, $x \neq 1$;
Range: $y \varepsilon R$

or D: $(-\infty, 1) \cup (1, \infty)$; R: $(-\infty, \infty)$

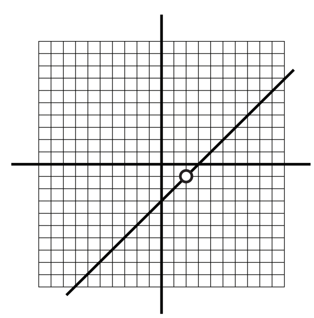
vi) Oblique Asymptote: $y = x + 3$



Example 7:

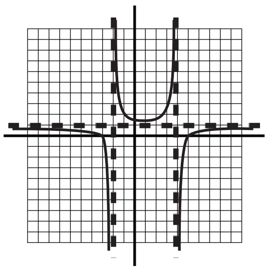
i) $y = x - 3$
ii) Hole: (2, -1)
iii) y - intercept: (0, -3)
iv) x - intercept(s): (3, 0)
v) Domain: $x \varepsilon R$, $x \neq 2$;
Range: $y \varepsilon R$, $y \neq -1$

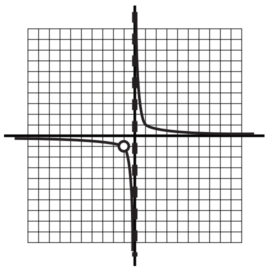
or D: $(-\infty, 2) \cup (2, \infty)$; R: $(-\infty, -1) \cup (-1, \infty)$



Example 8:

a) $y = \dfrac{(x+3)(x-5)}{(x+2)(x-4)}$ **b)** $y = \dfrac{x+1}{x(x+1)}$





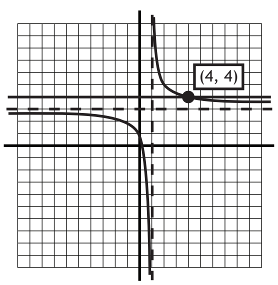
Example 9:

a) $y = \dfrac{x+1}{(x+4)(x-2)}$ **b)** $y = \dfrac{x(x+3)}{(x+2)(x+3)}$

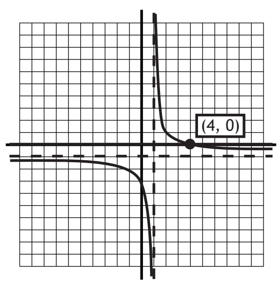
c) $y = \dfrac{7(x+6)(x+2)}{(x+6)(x+2)}$ **d)** $y = \dfrac{(x+3)(x+4)(x-6)}{(x+4)(x-6)}$

Example 10: **a)** x = 4

b)

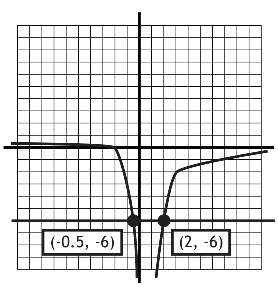


c)

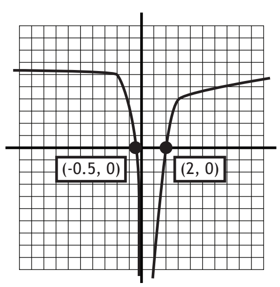


Example 11: **a)** x = -1/2 and x = 2

b)

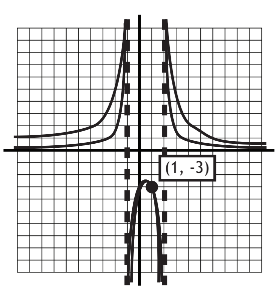


c)

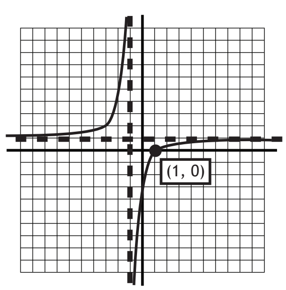


Example 12: **a)** x = 1. x = 2 is an extraneous root

b)



c)



Example 13:

a)

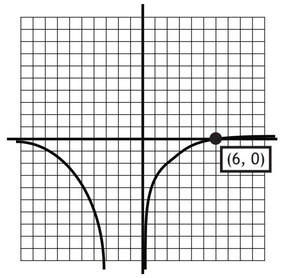
	d	*s*	*t*
Cynthia	15	x + 3	$\frac{15}{x+3}$
Alan	10	x	$\frac{10}{x}$

Equal times.

$$\frac{15}{x+3} = \frac{10}{x}$$

b) Cynthia: 9 km/h; Alan: 6 km/h

c) Graphing Solution: *x-intercept method*.



Example 14:

a)

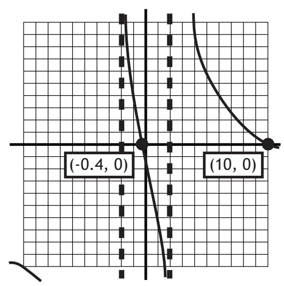
	d	*s*	*t*
Upstream	24	x - 2	$\frac{24}{x-2}$
Downstream	24	x + 2	$\frac{24}{x+2}$

Sum of times equals 5 h.

$$\frac{24}{x-2} + \frac{24}{x+2} = 5$$

b) Canoe speed: 10 km/h

c) Graphing Solution: *x-intercept method*.

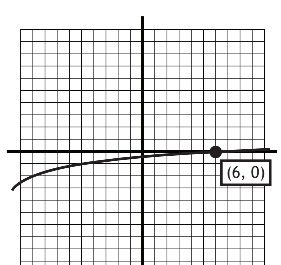


Example 15:

a) $0.40 = \dfrac{2+x}{14+x}$

b) Number of goals required: 6

c) Graphing Solution: *x-intercept method*.

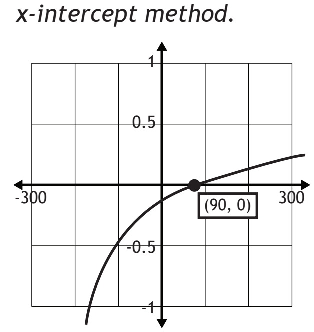


Example 16:

a) $0.50 = \dfrac{105+x}{300+x}$

b) Mass of almonds required: 90 g

c) Graphing Solution: *x-intercept method*.

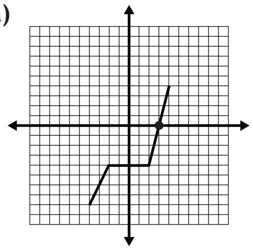


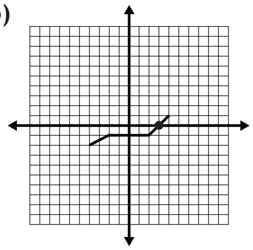
Answer Key

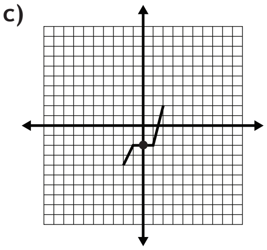
Transformations and Operations Lesson One: Basic Transformations

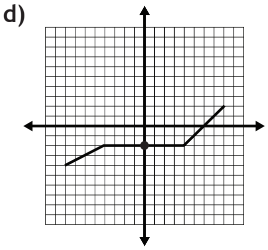
On this page,
● = invariant point

Example 1:

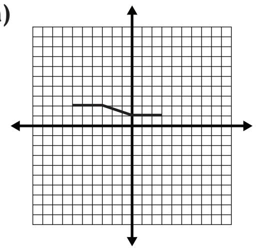
a) 

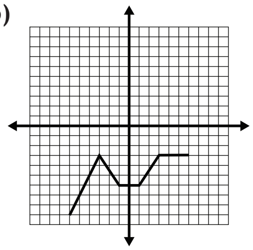
b) 

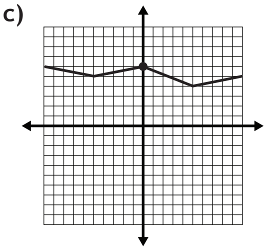
c) 

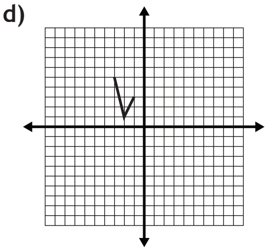
d) 

Example 2:

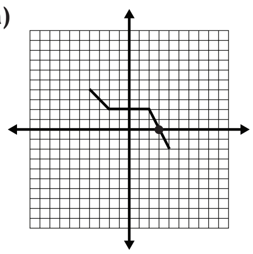
a) 

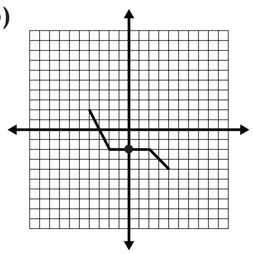
b) 

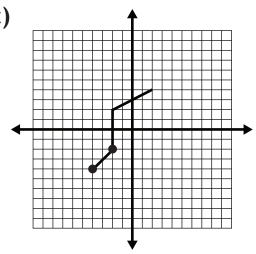
c) 

d) 

Example 3:

a) 

b) 

c) 

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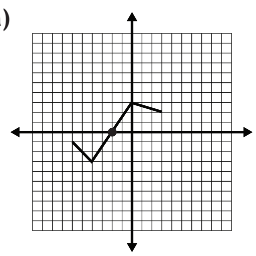
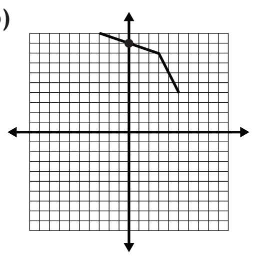
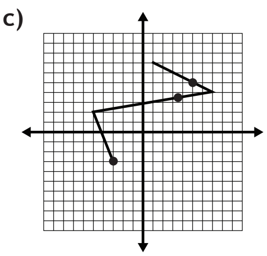
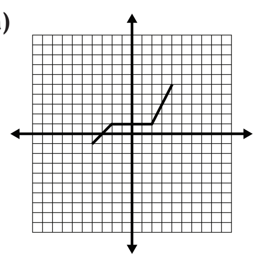
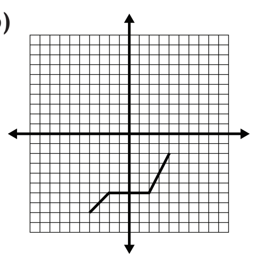
a) 

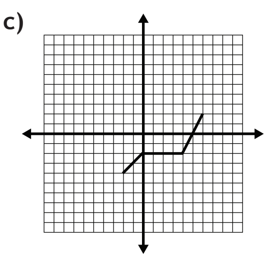
b) 

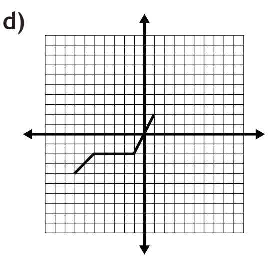
c) 

Example 5:

a) 

b) 

c) 

d) 

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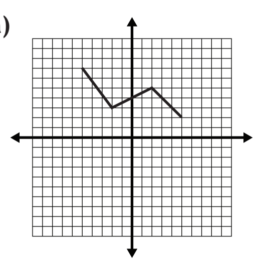
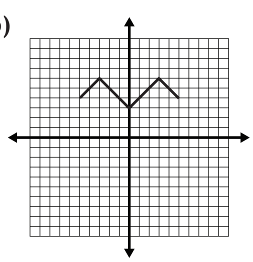
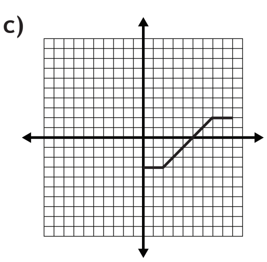
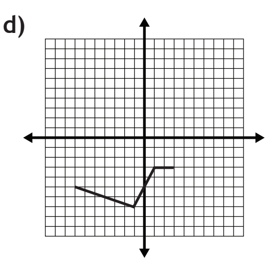
a) 

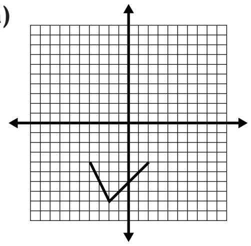
b) 

c) 

d) 

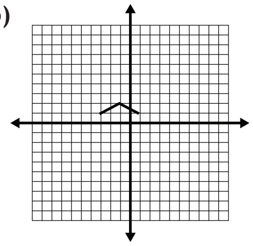
Answer Key

Example 7:

a)


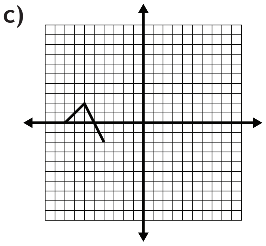
$y = f(2x)$

$(x,y) \to \left(\frac{1}{2}x, y\right)$

b)


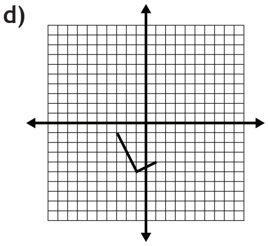
$y = f(x+6)$

$(x,y) \to (x-6, y)$

c)


$y = f(x) - 4$

$(x,y) \to (x, y-4)$

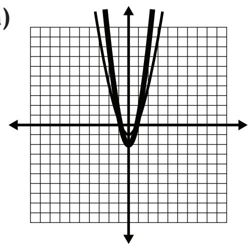
d)


$y = -f(x)$

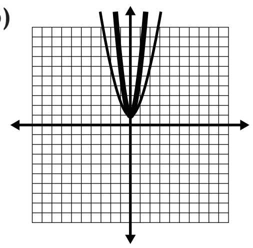
$(x,y) \to (x, -y)$

Example 8: a) $y = f(x) - 4$ b) $y = f(3x)$ c) $y = f\left(\frac{1}{2}x\right)$ d) $y = -f(x)$

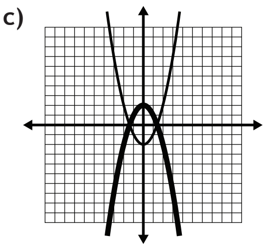
Example 9:

a)


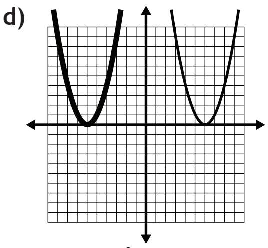
$y = 2x^2 - 2$

b)


$y = 4x^2 + 1$

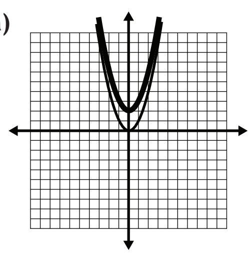
c)


$y = -x^2 + 2$

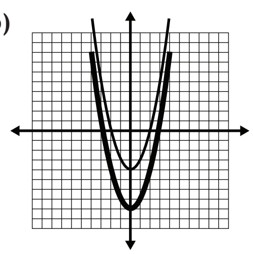
d)


$y = (-x-6)^2$

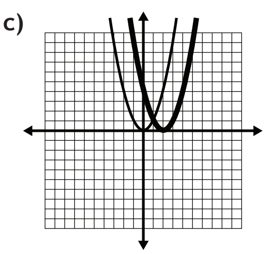
Example 10:

a)


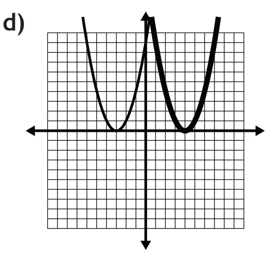
$y = x^2 + 2$

b)


$y = x^2 - 8$

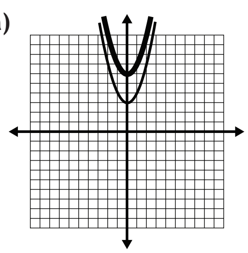
c)


$y = (x-2)^2$

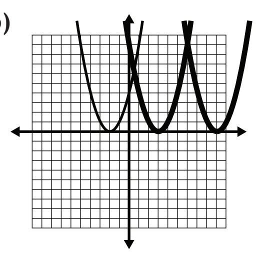
d)


$y = (x-4)^2$

Example 11:

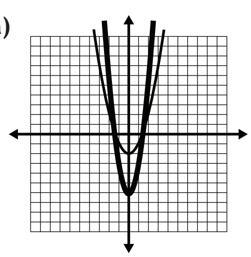
a)


$y = f(x) + 3$

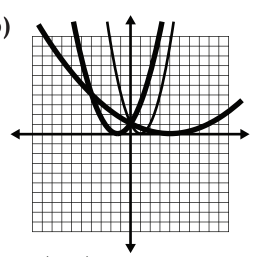
b)


$y = f(x-5)$ or $y = f(x-11)$

Example 12:

a)


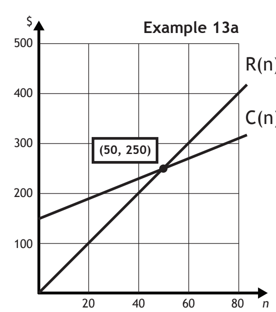
$y = 3f(x)$

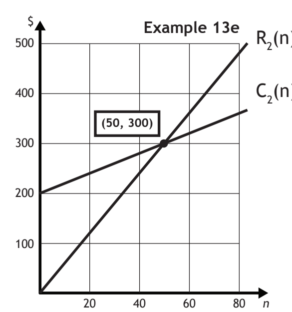
b)


$y = f\left(\frac{1}{4}x\right)$ or $y = f\left(-\frac{3}{4}x\right)$

Example 13:

a) R(n) = 5n
C(n) = 2n + 150
b) 50 loaves
c) C_2(n) = 2n + 200
d) R_2(n) = 6n
e) 50 loaves





Example 14:

a) $h(d-2) = -\frac{1}{9}(d-6)^2 + 4$

b) 12 metres

Answer Key

Transformations and Operations Lesson Two: Combined Transformations

Example 1: **a)** ***a*** is the vertical stretch factor.
b is the reciprocal of the horizontal stretch factor.
h is the horizontal displacement.
k is the vertical displacement.

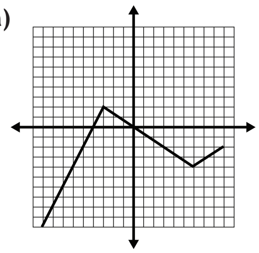
b) i. V.S. 1/3
H.S. 1/5

ii. V.S. 2
H.S. 4

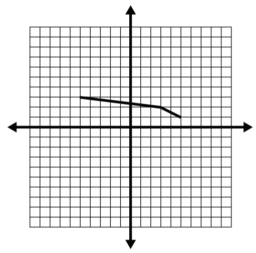
iii. V.S. 1/2
H.S. 3
Reflection about x-axis

iv. V.S. 3
H.S. 1/2
Reflection about x-axis
Reflection about y-axis

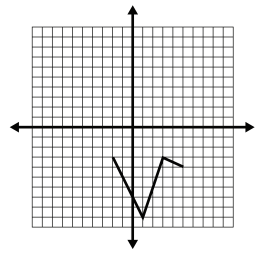
Example 2: **a)**



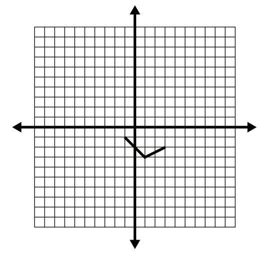
b)



c)



d)



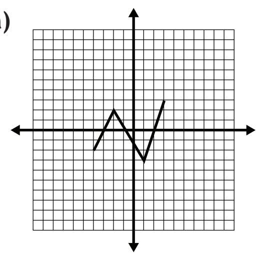
Example 3: **a)** H.T. 3 left **b) i.** H.T. 1 right
V.T. 3 up

ii. H.T. 2 left
V.T. 4 down

iii. H.T. 2 right
V.T. 3 down

iv. H.T. 7 left
V.T. 5 up

Example 4: **a)**



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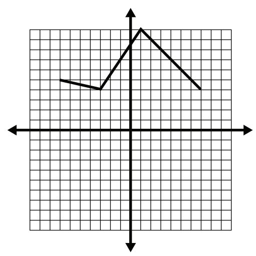

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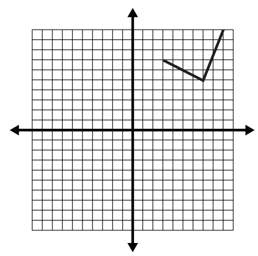
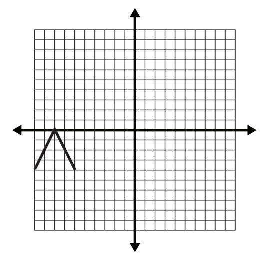


d)



Example 5: **a)** Stretches and reflections should be applied first, in any order. Translations should be applied last, in any order.

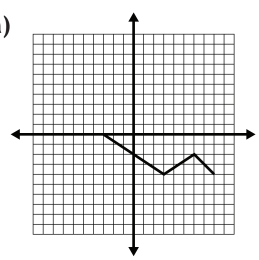
b) i. V.S. 2
H.T. 3 left
V.T. 1 up

ii. H.S. 3
Reflection about x-axis
V.T. 4 down

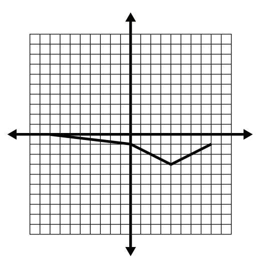
iii. V.S. 1/2
Reflection about y-axis
H.T. 2 left; V.T. 3 down

iv. V.S. 3; H.S. 1/4
Reflection about x-axis
Reflection about y-axis
H.T. 1 right; V.T. 2 up

Example 6: **a)**



b)



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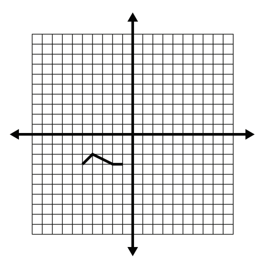

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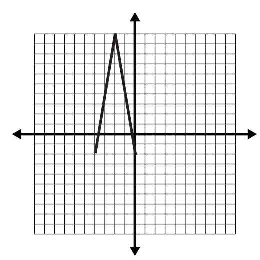
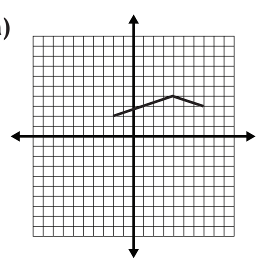


Example 7: **a)**



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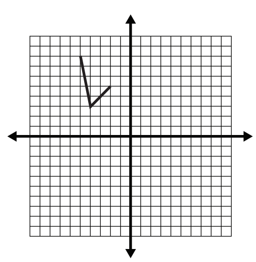
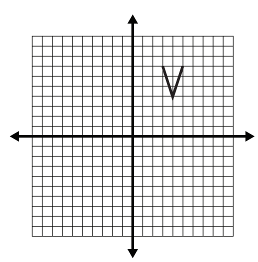


c)



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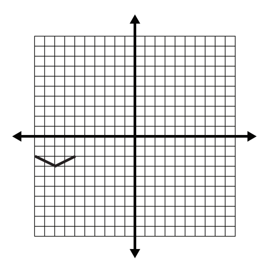
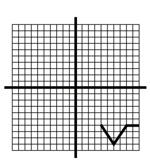
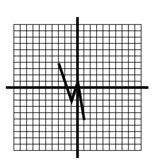


Example 8:
a) (1, 0)
b) (3, 6)
c) m = 8
and n = 1

Example 9:
a) y = -3f(x – 2)
b) y = -f[3(x + 2)]



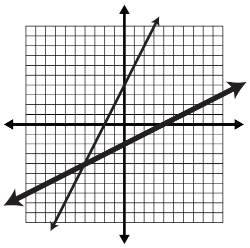


Example 10:
Axis-Independence
Apply all the vertical transformations together and apply all the horizontal transformations together, in either order.

Example 11:
a) H.T. 8 right; V.T. 7 up
b) Reflection about x-axis; H.T. 4 left; V.T. 6 down
c) H.S. 2; H.T. 3 left; V.T. 7 up
d) H.S. 1/2; Reflection about x & y-axis;
H.T. 5 right; V.T. 7 down.
e) The spaceship is not a function, and it must be translated in a specific order to avoid the asteroids.

Transformations and Operations Lesson Three: Inverses

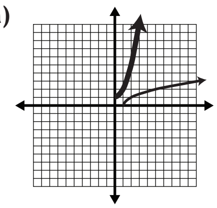
Example 1: a) Line of Symmetry: y = x



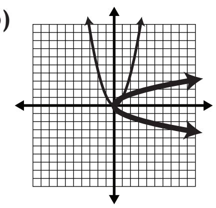
b) $f^{-1}(x) = \frac{1}{2}x - 2$

Example 2:

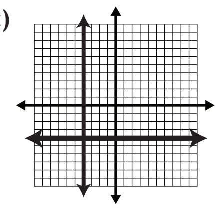
a)



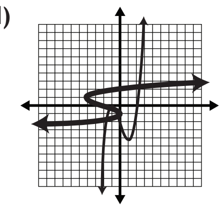
b)



c)

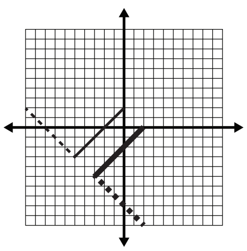


d)



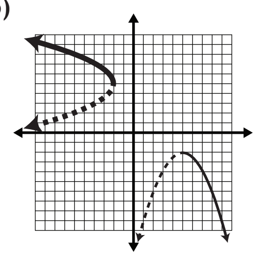
Example 3:

a)



Restrict the domain of the original function to -10 ≤ x ≤ -5 or -5 ≤ x ≤ 0

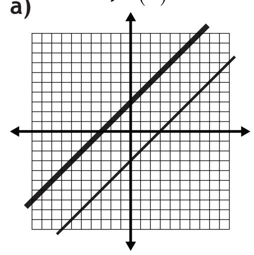
b)



Restrict the domain of the original function to x ≤ 5 or x ≥ 5.

Example 4:

a) $f^{-1}(x) = x + 3$

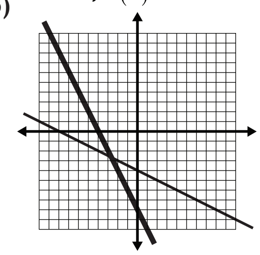


Original:
D: x ε R
R: y ε R

Inverse:
D: x ε R
R: y ε R

The inverse is a function.

b) $f^{-1}(x) = -2x - 8$



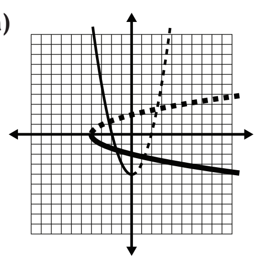
Original:
D: x ε R
R: y ε R

Inverse:
D: x ε R
R: y ε R

The inverse is a function.

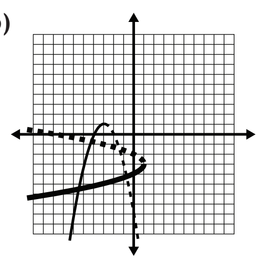
Example 5:

a)



Restrict the domain of the original function to x ≤ 0 or x ≥ 0

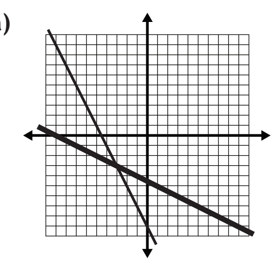
b)



Restrict the domain of the original function to x ≤ -3 or x ≥ -3.

Example 6:

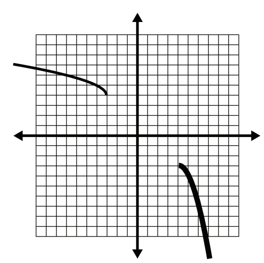
a)



$f^{-1}(x) = -\frac{1}{2}x - \frac{9}{2}$

D: x ε R

b)



$f^{-1}(x) = -(x-4)^2 - 3$

D: x ≥ 4

Example 7:

a) (10, 8)

b) True.
$f^{-1}(b) = a$

c) f(5) = 4

d) k = 30

Example 8:

a) 28 °C is equivalent to 82.4 °F

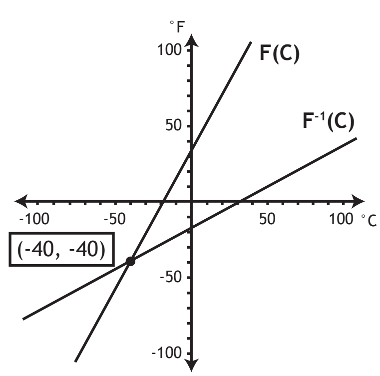
b) $C(F) = \frac{5}{9}F - \frac{160}{9}$

c) 100 °F is equivalent to 37.8 °C

d) C(F) can't be graphed since its dependent variable is C, but the dependent variable on the graph's y-axis is F. This is a mismatch.

e) $F^{-1}(C) = \frac{5}{9}C - \frac{160}{9}$

f) The invariant point occurs when the temperature in degrees Fahrenheit is equal to the temperature in degrees Celsius. -40 °F is equal to -40 °C.

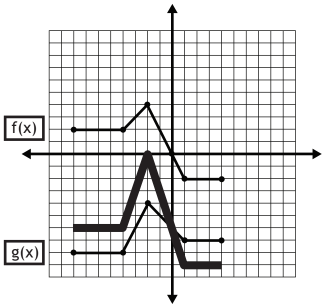


Answer Key

Transformations and Operations Lesson Four: Function Operations

Example 1: a)

x	(f + g)(x)
-8	-6
-4	-6
-2	0
0	-6
1	-9
4	-9

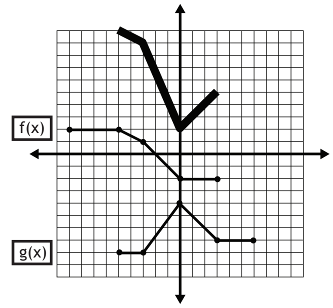


Domain:
$-8 \le x \le 4$
or [-8, 4]

Range:
$-9 \le y \le 0$
or [-9, 0]

b)

x	(f - g)(x)
-9	DNE
-5	10
-3	9
0	2
3	5
6	DNE

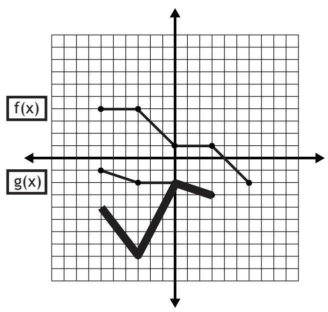


Domain:
$-5 \le x \le 3$;
or [-5, 3]

Range:
$2 \le y \le 10$
or [2, 10]

c)

x	(f • g)(x)
-6	-4
-3	-8
0	-2
3	-3
6	DNE

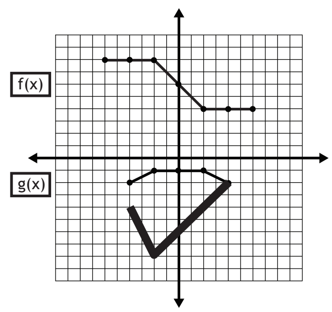


Domain:
$-6 \le x \le 3$
or [-6, 3]

Range:
$-8 \le y \le -2$
or [-8, -2]

d)

x	(f ÷ g)(x)
-6	DNE
-4	-4
-2	-8
0	-6
2	-4
4	-2
6	DNE



Domain:
$-4 \le x \le 4$
or [-4, 4]

Range:
$-8 \le y \le -2$
or [-8, -2]

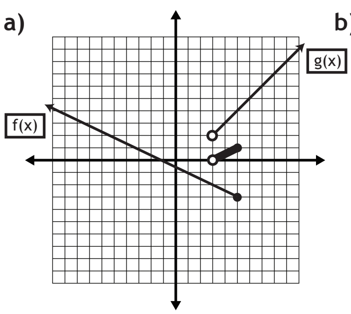
Example 2:

a) i. (f + g)(-4) = -2   ii. h(x) = -2; h(-4) = -2
b) i. (f - g)(6) = 8   ii. h(x) = 2x - 4; h(6) = 8
c) i. (fg)(-1) = -8   ii. h(x) = $-x^2 + 4x - 3$; h(-1) = -8
d) i. (f/g)(5) = -0.5   ii. h(x) = (x - 3)/(-x + 1); h(5) = -0.5

> **Reminder:** Math 30-1 students are expected to know that domain and range can be expressed using ***interval notation***.

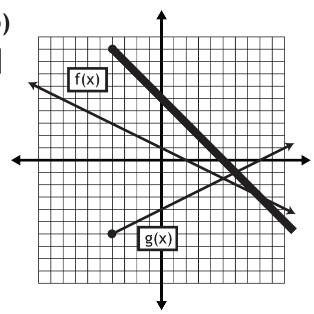
Example 3:

a)



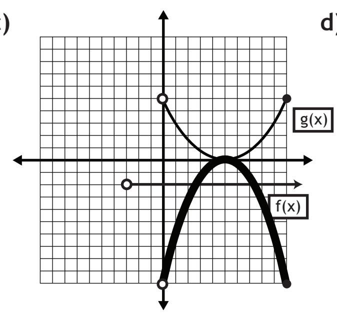
Domain: $3 < x \le 5$ or (3, 5]
Range: $0 < y \le 1$ or (0, 1]

b)



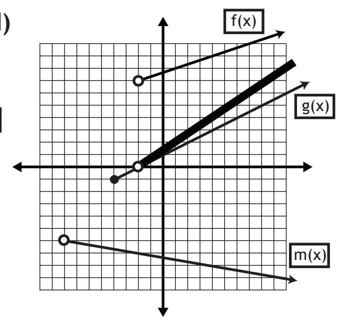
Domain: $x \ge -4$ or [-4, ∞)
Range: $y \le 9$ or (-∞, 9]

c)



Domain: $0 < x \le 10$ or (0, 10]
Range: $-10 \le y \le 0$ or [-10, 0]

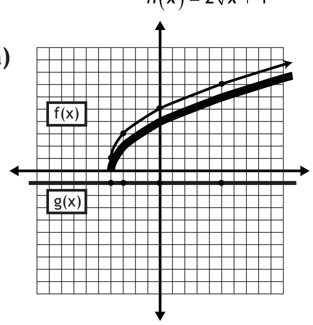
d)



Domain: $x > -2$ or (-2, ∞)
Range: $y > 0$ or (0, ∞)

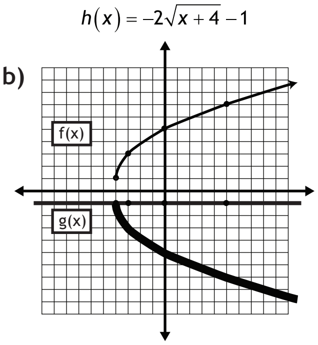
Example 4:

a) $h(x) = 2\sqrt{x+4}$



Domain: $x \ge -4$ or [-4, ∞)
Range: $y \ge 0$ or [0, ∞)
Transformation: y = f(x) - 1

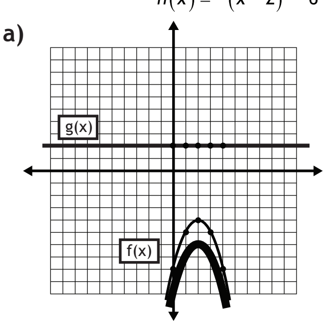
b) $h(x) = -2\sqrt{x+4} - 1$



Domain: $x \ge -4$ or [-4, ∞)
Range: $y \le -1$ or (-∞, -1]
Transformation: y = -f(x).

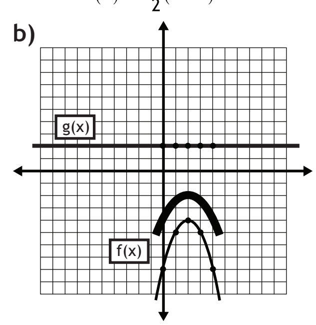
Example 5:

a) $h(x) = -(x-2)^2 - 6$



Domain: x ε R or (-∞, ∞)
Range: $y \le -6$ or (-∞, -6]
Transformation: y = f(x) - 2

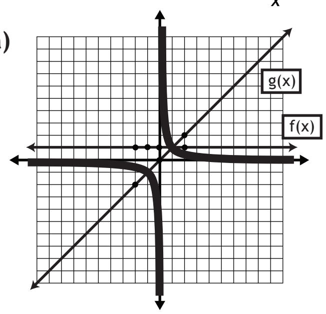
b) $h(x) = -\frac{1}{2}(x-2)^2 - 2$



Domain: x ε R or (-∞, ∞)
Range: $y \le -2$ or (-∞, -2]
Transformation: y = 1/2f(x)

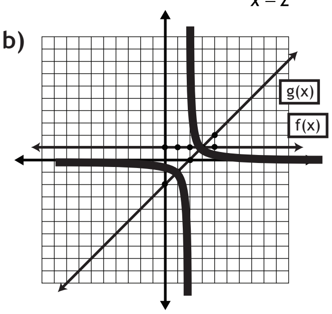
Example 6:

a) $h(x) = \dfrac{1}{x}$



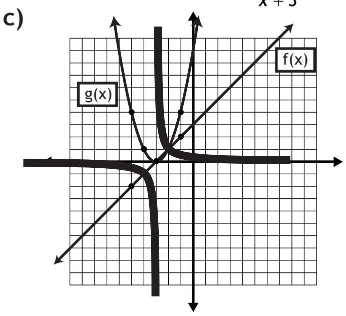
Domain: x ε R, x ≠ 0;
Range: y ε R, y ≠ 0
or D: (-∞, 0) U (0, ∞); R: (-∞, 0) U (0, ∞)

b) $h(x) = \dfrac{1}{x-2}$



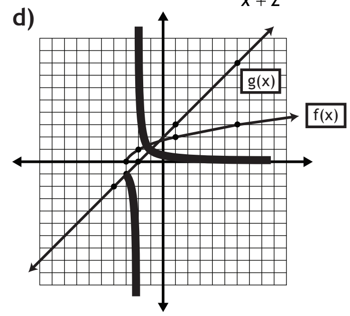
Domain: x ε R, x ≠ 2;
Range: y ε R, y ≠ 0
or D: (-∞, 2) U (2, ∞); R: (-∞, 0) U (0, ∞)

c) $h(x) = \dfrac{1}{x+3}$



Domain: x ε R, x ≠ -3;
Range: y ε R, y ≠ 0
or D: (-∞, -3) U (-3, ∞); R: (-∞, 0) U (0, ∞)

d) $h(x) = \dfrac{\sqrt{x+3}}{x+2}$



Domain: x ≥ -3, x ≠ -2;
Range: y ε R, y ≠ 0
or D: [-3, -2) U (-2, ∞); R: (-∞, 0) U (0, ∞)

Example 7:

a) $A_L(x) = 8x^2 - 8x$

b) $A_S(x) = 3x^2 - 3x$

c) $A_L(x) - A_S(x) = 10$; $x = 2$

d) $A_L(2) + A_S(2) = 22$;

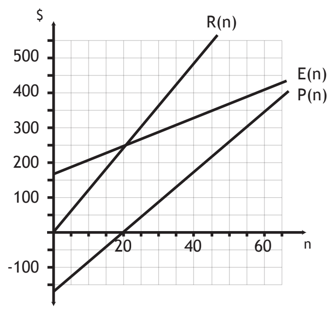
e) The large lot is 2.67 times larger than the small lot

Example 8:

a) R(n) = 12n;
E(n) = 4n + 160;
P(n) = 8n – 160

b) When 52 games are sold, the profit is $256

c) Greg will break even when he sells 20 games



Example 9:

a) The surface area and volume formulae have two variables, so they may not be written as single-variable functions.

b) $h = \sqrt{3}r$ **c)** $s = 2r$ **d)** $SA(r) = 3\pi r^2$

$V(r) = \dfrac{\sqrt{3}}{3}\pi r^3$

e) $\dfrac{SA}{V}(r) = \dfrac{3\sqrt{3}}{r}$ **f)** $\dfrac{SA}{V}(6) = \dfrac{\sqrt{3}}{2}$

Transformations and Operations Lesson Five: Function Composition

Example 1: **a)**

x	*g(x)*	*f(g(x))*
-3	9	6
-2	4	1
-1	1	-2
0	0	-3
1	1	-2
2	4	1
3	9	6

b)

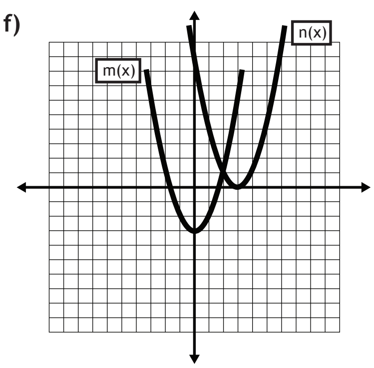
x	*f(x)*	*g(f(x))*
0	-3	9
1	-2	4
2	-1	1
3	0	0

c) Order matters in a composition of functions.

d) $m(x) = x^2 - 3$

e) $n(x) = (x - 3)^2$

f)

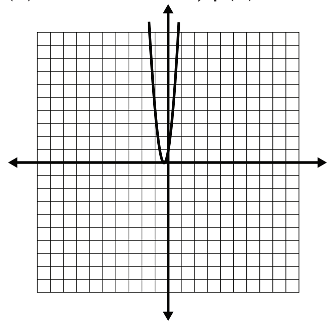


Example 2: **a)** m(3) = 33 **b)** n(1) = -4 **c)** p(2) = -2 **d)** q(-4) = -16

Example 3: **a)** $m(x) = 4x^2 - 3$ **b)** $n(x) = 2x^2 - 6$ **c)** $p(x) = x^4 - 6x^2 + 6$ **d)** q(x) = 4x **e)** All of the results match

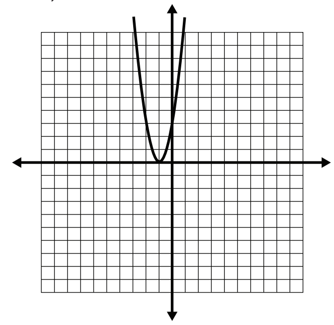
Example 4: **a)** $m(x) = (3x + 1)^2$

The graph of f(x) is horizontally stretched by a scale factor of 1/3.



b) $n(x) = 3(x + 1)^2$

The graph of f(x) is vertically stretched by a scale factor of 3.

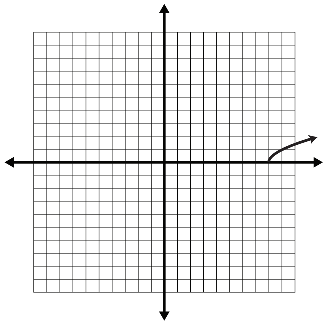


Answer Key

Example 5: **a)** $m(x) = \sqrt{x-8}$

Domain:
x ≥ 8 or [8, ∞)

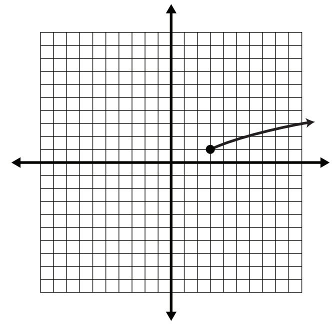
A composite function can only exist over a domain where both component functions exist



b) $m(x) = \sqrt{x-2}$

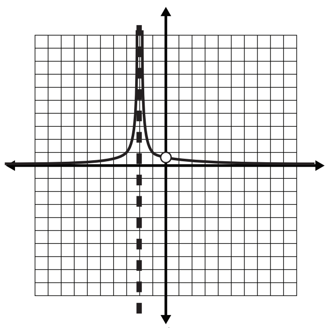
Domain:
x ≥ 3 or [3, ∞)

A composite function can only exist over a domain where both component functions exist



Example 6: **a)** $h(x) = \frac{1}{|x+2|}$

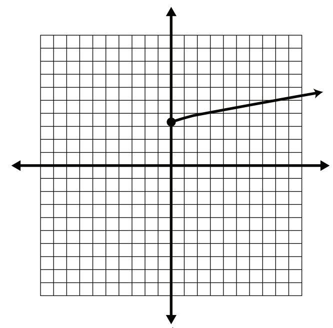
Domain:
x ε R, x ≠ -2, 0
or D: (-∞, -2) U (-2, 0) U (0, ∞)



b) $h(x) = \sqrt{x+2} + 2$

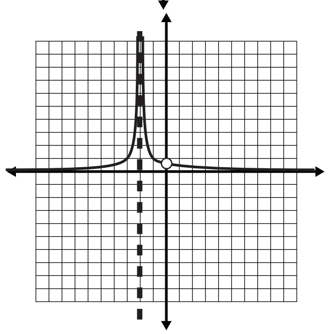
Domain:
x ε R, x ≥ 0 or [0, ∞)

Since the component function f(x) has a domain of x ≥ 0, the domain of h(x) can't have any values less than zero.



Example 7: **a)** $h(x) = \frac{1}{(x+2)^2}$

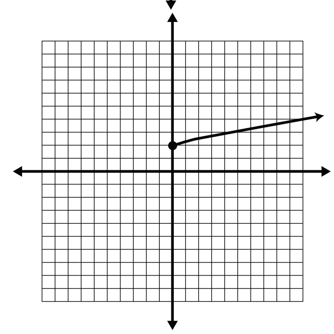
Domain:
x ε R, x ≠ -2, 0.
or D: (-∞, -2) U (-2, 0) U (0, ∞)



b) $h(x) = \sqrt{2x+4}$

Domain:
x ε R, x ≥ 0 or [0, ∞)

Since the component function f(x) has a domain of x ≥ 0, the domain of h(x) can't have any values less than zero.



Example 8: **a)** $f(x) = 2x;\ \ g(x) = x+1$ **b)** $f(x) = \frac{1}{x};\ \ g(x) = x^2 - 1$ **c)** $f(x) = x^2 - 5x + 1;\ \ g(x) = x+1$

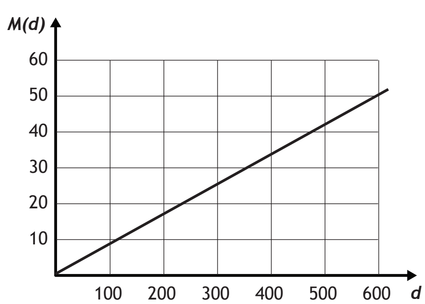
d) $f(x) = x^2;\ \ g(x) = x+2$ **e)** $f(x) = 2\sqrt{x};\ \ g(x) = \frac{1}{x}$ **f)** $f(x) = \sqrt{x};\ \ g(x) = x^2$

Example 9:

a) $(f^{-1} \circ f)(x) = x$, so the functions are inverses of each other.
b) $(f^{-1} \circ f)(x) \neq x$, so the functions are NOT inverses of each other.

Example 10:

a) The cost of the trip is $4.20. It took two separate calculations to find the answer.
b) V(d) = 0.08d
c) M(V) = 1.05V
d) M(d) = 0.084d
e) Using function composition, we were able to solve the problem with one calculation instead of two.



Example 11:

a) $A(t) = 900\pi t^2$

b) $A = 8100\pi\ cm^2$

c) t = 7 s; r = 210 cm

Example 12:

a) a(c) = 1.03c

b) j(a) = 78.0472a

c) b(a) = 0.6478a

d) b(c) = 0.6672c

Example 13:

a) $r(h) = \frac{3h}{8}$

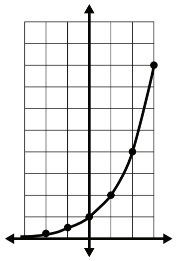
b) $V_{water}(h) = \frac{3}{64}\pi h^3$

c) h = 4 cm

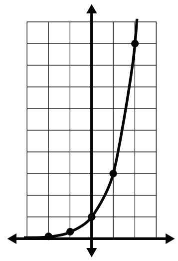
Answer Key

Exponential and Logarithmic Functions Lesson One: Exponential Functions

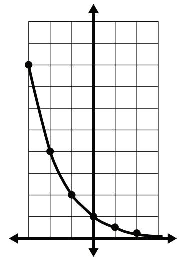
Example 1: a)



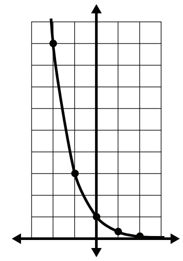
b)



c)



d)

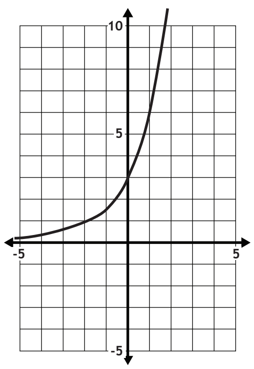


Parts (a-d):
Domain: x ε R or (-∞, ∞)
Range: y > 0 or (0, ∞)
x-intercept: None
y-intercept: (0, 1)
Asymptote: y = 0

An exponential function is defined as $y = b^x$, where $b > 0$ and $b \neq 1$. When $b > 1$, we get exponential growth. When $0 < b < 1$, we get exponential decay. Other b-values, such as -1, 0, and 1, will not form exponential functions.

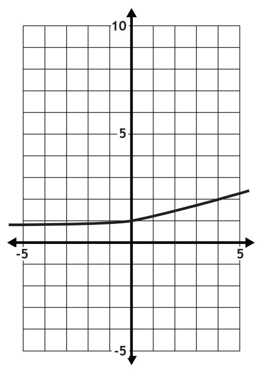
Example 2: a) $f(x) = 4^x$; $n = \frac{1}{16}$ b) $f(x) = \left(\frac{3}{2}\right)^x$; $n = \frac{8}{27}$ c) $f(x) = \left(\frac{1}{5}\right)^x$; $n = \frac{1}{5}$ d) $f(x) = \left(\frac{3}{4}\right)^x$; $n = \frac{27}{64}$

Example 3: a)



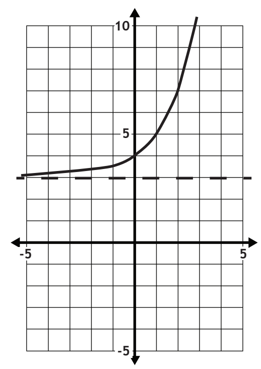
Domain: x ε R or (-∞, ∞)
Range: y > 0 or (0, ∞)
Asymptote: y = 0

b)



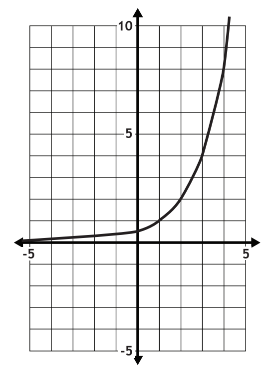
Domain: x ε R or (-∞, ∞)
Range: y > 0 or (0, ∞)
Asymptote: y = 0

c)



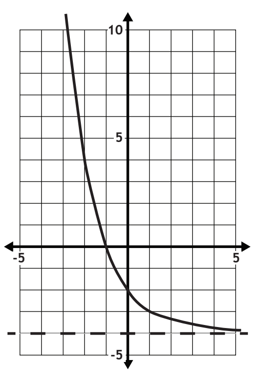
Domain: x ε R or (-∞, ∞)
Range: y > 3 or (3, ∞)
Asymptote: y = 3

d)



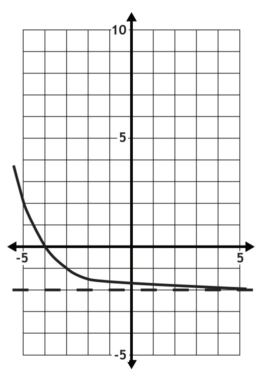
Domain: x ε R or (-∞, ∞)
Range: y > 0 or (0, ∞)
Asymptote: y = 0

Example 4: a)



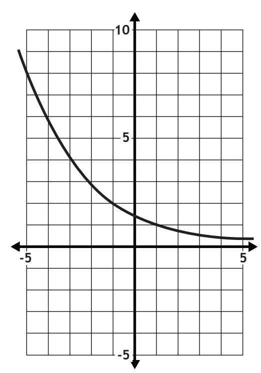
Domain: x ε R or (-∞, ∞)
Range: y > -4 or (-4, ∞)
Asymptote: y = -4

b)



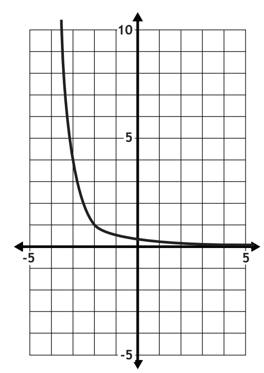
Domain: x ε R or (-∞, ∞)
Range: y > -2 or (-2, ∞)
Asymptote: y = -2

c)



Domain: x ε R or (-∞, ∞)
Range: y > 0 or (0, ∞)
Asymptote: y = 0

d)



Domain: x ε R or (-∞, ∞)
Range: y > 0 or (0, ∞)
Asymptote: y = 0

Example 5: a) $f(x) = \left(\frac{2}{3}\right)^x - 3$

$n = \frac{147}{32}$

b) $f(x) = \frac{1}{4}(3)^x + 1$

$n = \frac{973}{972}$

Example 6: a) $\left(0, \frac{a}{b^4}\right)$ b) $a = \frac{25}{3}$ c) $y = \frac{3}{4}\left(\frac{1}{3}\right)^x$ d) $y = 2^x - 3$ e) V.S. of 9 equals H.T. 2 units left. f) See Video

Answer Key

Example 7:

a) $x = 2$
b) $x = 16$
c) $x = \frac{1}{243}$
d) $x = \frac{1}{2}$

Example 8:

a) $x = 4$
b) $x = 5$
c) $x = 5$
d) $x = 6$
e) $x = -2;\ y = \frac{7}{2}$
f) $m = -\frac{11}{6};\ n = -3$

Example 9:

a) $x = -2$
b) $x = 4$
c) $x = 1$
d) $x = \frac{1}{2}$

Example 10:

a) $x = 6$
b) $x = 5$
c) $x = 18$
d) $x = 15$

Example 11:

a) $x = 2$
b) *infinite solutions*
c) $x = 1$
d) $x = 3$

Example 12:

a) $x = \frac{5}{2}$
b) $x = \frac{7}{2}$
c) $x = 3$
d) $x = 1$

Example 13:

a) $x = \frac{1}{2}, 1$
b) $x = -2, 1$
c) $x = 2$
d) $x = 5$

Example 14:

a) $x = 1.77$
b) *no solution*
c) $x = 1.79$
d) $x = 3$

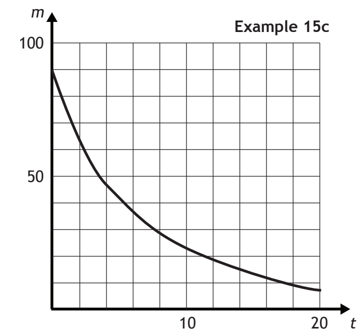
Example 15:

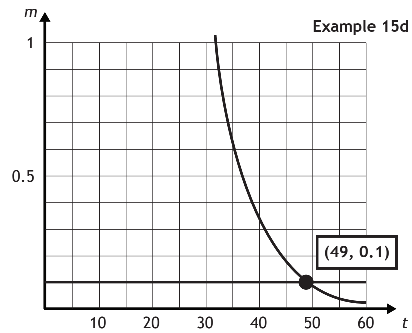
a) $m(t) = 90\left(\frac{1}{2}\right)^{\frac{t}{5}}$

b) 84 g

c) See Graph

d) 49 years





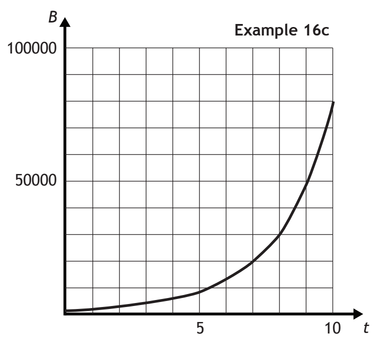
Example 16:

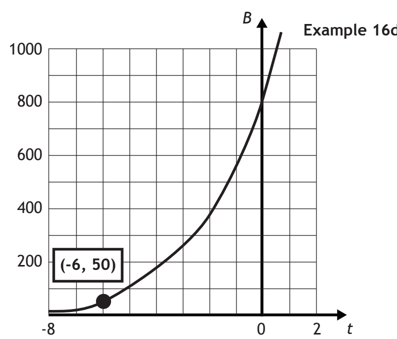
a) $B(t) = 800(2)^{\frac{t}{90}}$

b) 32254 bacteria

c) See Graph

d) 6 hours ago





Watch Out! The graph requires hours on the t-axis, so we can rewrite the exponential function as:

$$B(t) = 800(2)^{\frac{t}{1.5}}$$

Example 17: **a)** $S(t) = 16(1.44)^t$; 69 MHz **b)** $C(t) = 2500(0.70)^t$; \$600

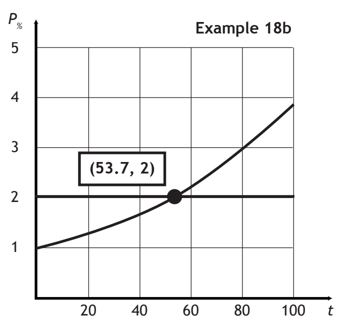
Example 18:

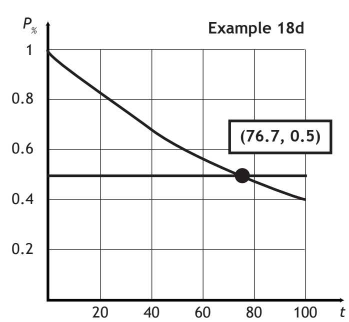
a) 853,370

b) 54 years

c) 21406

d) 77 years





Example 19:

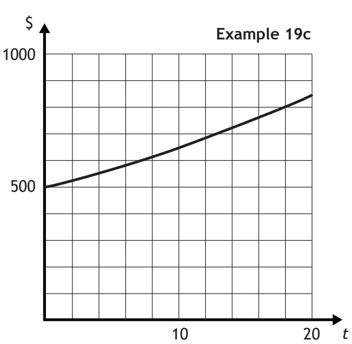
a) $A(t) = 500(1.025)^t$

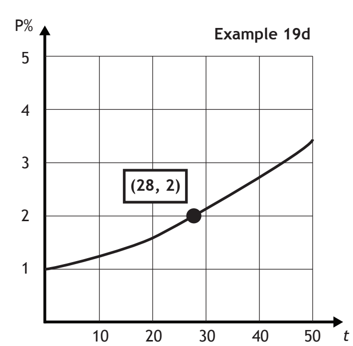
b) \$565.70
Interest: \$65.70

c) See graph

d) 28 years

e) \$566.14; \$566.50; \$566.57
As the compounding frequency increases, there is less and less of a monetary increase.





Answer Key

Exponential and Logarithmic Functions Lesson Two: Laws of Logarithms

Example 1:

a) The base of the logarithm is ***b***, ***a*** is called the argument of the logarithm, and ***E*** is the result of the logarithm.

In the exponential form, ***a*** is the result, ***b*** is the base, and ***E*** is the exponent.

b) i. 0; 1; 2; 3 ii. 0; 1; 2; 3

c) i. $\log_4 2$ ii. $\log_9\left(\frac{1}{3}\right)$

Example 2:

a) $\log_9\left(\frac{1}{3}\right)$, $\log_{16}\left(\frac{1}{2}\right)$, $\log_5 1$, $\log 10$, $\log_2 16$

b) $\log_{\frac{1}{3}} 27$, $\log_{\frac{1}{4}} 8$, $\log_{\frac{1}{8}}\left(\frac{1}{2}\right)$, $\log_{\frac{1}{4}}\left(\frac{1}{2}\right)$, $\log_{\frac{1}{8}}\left(\frac{1}{8}\right)$

c) $\log_8 3$, $\log_6 7$, $\log_{\frac{1}{4}}\left(\frac{1}{15}\right)$, $\log_3 25$

Example 3:

a) $y = 2^x$

b) $y = 16$

c) $y = 10^{\frac{x}{a}}$

d) $y = \frac{3^x}{2}$

e) $y = x^{\frac{1}{2}}$

f) $y = 8 + x$

g) $y = (x+1)^2 - 1$

h) $y = 3^{2x-1}$

Example 4:

a) $\log_x y = 2$

b) $\log_x \frac{y}{10} = 4$

c) $\log_{\frac{1}{3}} y = x$

d) $\log_x 3y = \frac{1}{2}$

e) $\log_{\frac{x}{2}} y = \frac{1}{3}$

f) $\log_{x-3} y = 2$

g) $\log_k y = x - 1$

h) $\log a = y - x$

Example 5:

a) 3

b) 3

c) 2

d) 2

e) $\log_{25} 5$

f) $\log_3 \sqrt{3}$

g) $\log_{\frac{1}{3}} \frac{1}{2}$

h) $\log_a b$

Example 6:

a) $\log x + \log y$

b) can't expand

c) $\log 3 + \log(x+1)$

d) $1 + \log x$

e) $\log 12$

f) $\log \frac{1}{2}$

g) $\log x^5$

h) $\log(x^2 - x - 2)$

Example 7:

a) $\log x - \log y$

b) can't expand

c) $\log(x+1) - 2$

d) $\log_3 x - 1 - \log_3(x+1)$

e) $\log 3$

f) $\log \frac{1}{6}$

g) $\log x^3$

h) $\log\left(\frac{2x}{x+3}\right)$

Example 8:

a) $2\log x$

b) can't expand

c) $7\log x$

d) $a\log x + \log x$

e) $\log x^3$

f) $\log(x-1)^2$

g) $\log(8x^6)$

h) $\log x^2$

Example 9:

a) *undefined*

b) *undefined*

c) 0

d) 1

e) x

f) x

g) $2k$

h) $\frac{k}{2}$

Example 10:

a) 160

b) $4a$

c) $2k$

d) $\frac{h}{2}$

e) –7

f) 7

g) $\log(10x)$

h) $\log_2(8x)$

Example 11:

a) $x = \log_3 4$

b) no solution

c) $x = \log_5\left(\frac{7}{2}\right) - 2$

d) $x = \log_{\frac{2}{5}}\left(\frac{1}{3}\right) + 3$

Example 12:

a) $x = \frac{-\log 3}{5\log 6 - 2\log 3}$

b) $x = \frac{-\log 3 - 3\log 2}{\log 2 - 2\log 3}$

c) $x = \frac{\log 4 + \log 3}{2\log 4 - \log 5}$

d) $x = \frac{-\log 2 - 3\log 3}{\log 3 - 3\log 6}$

Example 13:

a) $x = 10$

b) $x = 8$

c) $x = -2$

d) $x = 14$

Example 14:

a) $x = 2$

b) $x = 5$

c) $x = \frac{2}{3}$

d) $x = \pm\sqrt{29}$

Example 15:

a) $x = 6$

b) $x = 5$

c) $x = \frac{1}{10}, 100000$

d) $x = \frac{1}{100}, 100$

Example 16:

a) $\frac{1}{4}$

b) $\log\left(\frac{\sqrt{a}}{b^3c^2}\right)$

c) 16

d) 2

e) $\log_b(2a) = \frac{5}{4}$

f) $\log_5 x$

g) $2x + 24$

h) $\log_3\left(9\sqrt[3]{x}\right)$

Example 17:

a) 12

b) 14

c) 3^{233}

d) 1

e) 100

f) $\log_2\left(a\sqrt{b}\right)$

g) 3

h) 15

Example 18:

a) –3

b) no solution

c) x

d) 1,100

e) a^2

f) $\frac{1}{2}$

g) see video

h) $\log_2(4x+2)$

Example 19:

a) 4

b) $\log(ab)^3$

c) 2

d) $\log(a+1)$

e) –2

f) $x = \pm 1$

g) $\frac{5}{2}$

h) $\log_{16}(4x^3)$

Example 20:

a) 2

b) $\frac{3\log 5 + \log 2}{3\log 2 - 2\log 5}$

c) 199

d) $\log\left(\frac{1}{x}\right)$

e) 9

f) $10^{\frac{8}{5}}$

g) $\log\left(\frac{a^4c}{b^{\frac{1}{2}}}\right)$

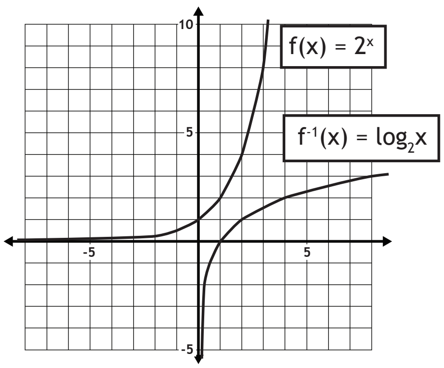
h) 2

Answer Key

Exponential and Logarithmic Functions Lesson Three: Logarithmic Functions

Example 1:

a) See Graph **b)** See Graph **c)** See Video



d)

	$y = 2^x$	$y = \log_2 x$
Domain	x ε R	x > 0
Range	y > 0	y ε R
x-intercept	none	(1, 0)
y-intercept	(0, 1)	none
Asymptote Equation	y = 0	x = 0

e)
i) -1,
ii) 0,
iii) 1,
iv) 2.8

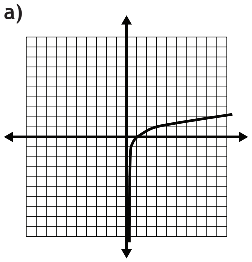
f)
$y = \log_1 x$, $y = \log_0 x$, and $y = \log_{-2} x$ are not functions.
$y = \log_{\frac{1}{10}} x$ is a function.

g) The logarithmic function $y = \log_b x$ is the inverse of the exponential function $y = b^x$. It is defined for all real numbers such that b>0 and x>0.

h) Graph $\log_2 x$ using logx/log2

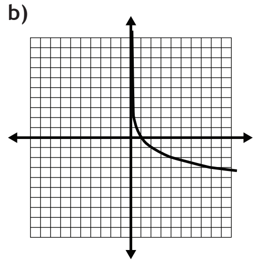
Example 2:

a)



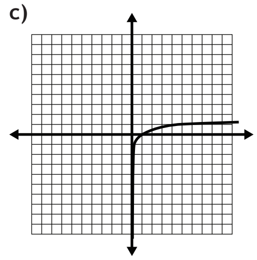
D: x > 0 or (0, ∞)
R: y ε R or (-∞, ∞)
A: x = 0

b)



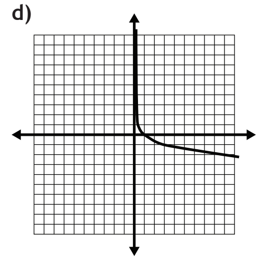
D: x > 0 or (0, ∞)
R: y ε R or (-∞, ∞)
A: x = 0

c)



D: x > 0 or (0, ∞)
R: y ε R or (-∞, ∞)
A: x = 0

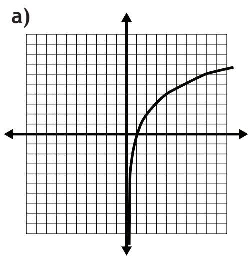
d)



D: x > 0 or (0, ∞)
R: y ε R or (-∞, ∞)
A: x = 0

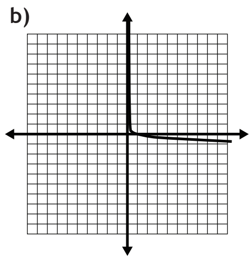
Example 3:

a)



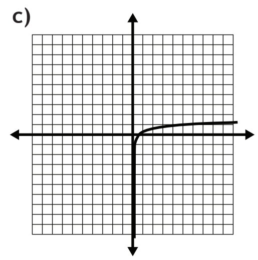
D: x > 0 or (0, ∞)
R: y ε R or (-∞, ∞)
A: x = 0

b)



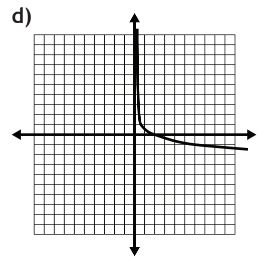
D: x > 0 or (0, ∞)
R: y ε R or (-∞, ∞)
A: x = 0

c)



D: x > 0 or (0, ∞)
R: y ε R or (-∞, ∞)
A: x = 0

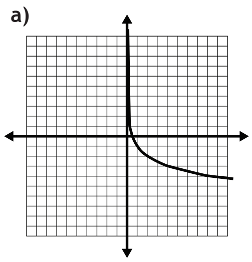
d)



D: x > 0 or (0, ∞)
R: y ε R or (-∞, ∞)
A: x = 0

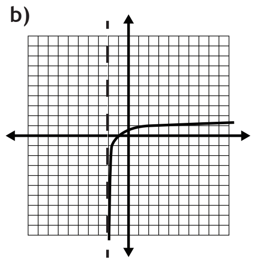
Example 4:

a)



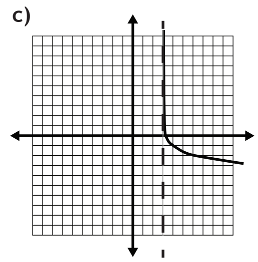
D: x > 0 or (0, ∞)
R: y ε R or (-∞, ∞)
A: x = 0

b)



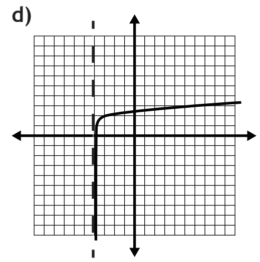
D: x > -2 or (-2, ∞)
R: y ε R or (-∞, ∞)
A: x = -2

c)



D: x > 3 or (3, ∞)
R: y ε R or (-∞, ∞)
A: x = 3

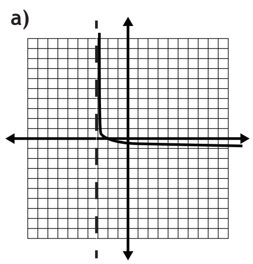
d)



D: x > -4 or (-4, ∞)
R: y ε R or (-∞, ∞)
A: x = -4

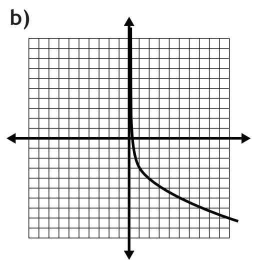
Example 5:

a)



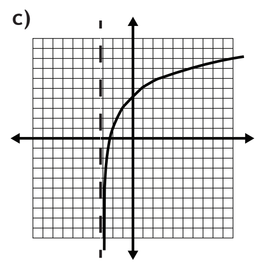
D: x > -3 or (-3, ∞)
R: y ε R or (-∞, ∞)
A: x = -3

b)



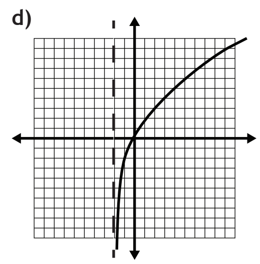
D: x > 0 or (0, ∞)
R: y ε R or (-∞, ∞)
A: x = 0

c)



D: x > -3 or (-3, ∞)
R: y ε R or (-∞, ∞)
A: x = -3

d)

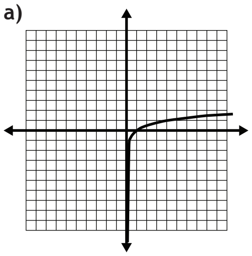


D: x > -2 or (-2, ∞)
R: y ε R or (-∞, ∞)
A: x = -2

Answer Key

Example 6:

a)

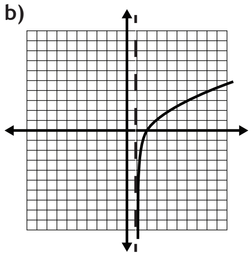


D: x > 0
or (0, ∞)

R: y ε R
or (-∞, ∞)

A: x = 0

b)

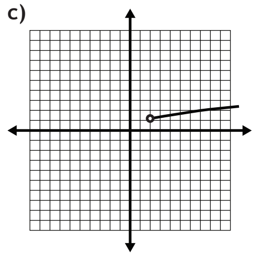


D: x > 1
or (1, ∞)

R: y ε R
or (-∞, ∞)

A: x = 1

c)

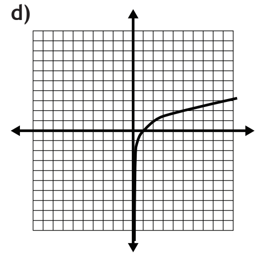


D: x > 2
or (2, ∞)

R: y > $\log_3 4$
or ($\log_3 4$, ∞)

A: none

d)



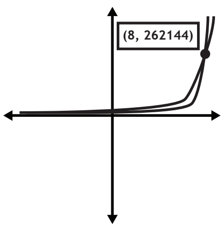
D: x > 0
or (0, ∞)

R: y ε R
or (-∞, ∞)

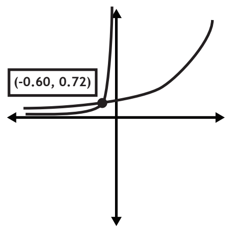
A: x = 0

Example 7:

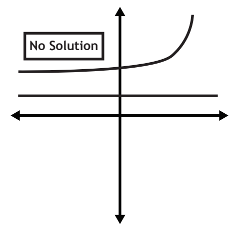
a) x = 8



b) $x = \dfrac{-2\log 5}{4\log 5 - \log 3}$

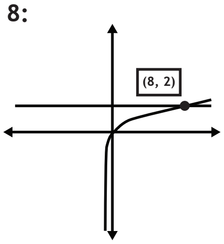


c) No Solution

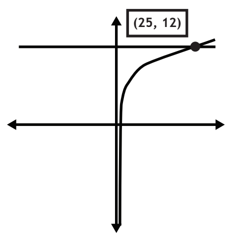


Example 8:

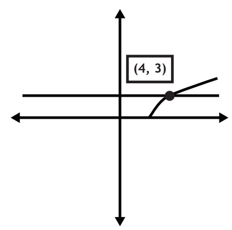
a) x = 8



b) x = 25



c) x = 4



Example 9:

a) $b = 2\sqrt{2}$

b) (-3, 0) and (0, 2)

c) $x = \dfrac{8}{3}$

d) $k = 81$

e) $0 < x < 6,\ x \neq 1$

Example 10:

a) $y = f(9x)$; $y = f(x) + 2$

b) $(-1, 1)$

c) $y = f(x) - 2$

d) $(1, 0)$

e) $x = \dfrac{1}{k}$

Example 11:

a) $x = 0$ *(y-axis)*

b) $g(x) = f^{-1}(x)$

c) $f^{-1}(x) = \log_3(x - 4)$

d) $x > 4$

e) $k = 8$

Example 12:

a) 4

b) 0.1 m

c) 31.6 times stronger

d) See Video

e) 10 times stronger

f) See Video

g) 5.5

h) 5.4

Example 13:

a) 60 dB

b) 0.1 W/m^2

c) 100 times more intense

d) See Video

e) 100 times more intense

f) See Video

g) 23 dB

h) 37 dB

Example 14:

a) pH = 4

b) 10^{-11} mol/L

c) 1000 times stronger

d) See Video

e) pH = 2

f) pH = 5

g) 100 times more acidic

Example 15:

a) 200 cents

b) 784 Hz

c) 1200 cents separate the two notes